MODULAR NAHM SUMS FOR SYMMETRIZABLE MATRICES OF INDICES $(2,\ldots,2,1)$ AND $(1,\ldots,1,2)$

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Abstract. In this paper, we present three families of modular Nahm sums for symmetrizable matrices with arbitrary rank $r \geq 2$ of indices $(2,\ldots,2,1)$ and $(1,\ldots,1,2)$. Specifically, the cases corresponding to $r=2$ and $r=3$ of these families have been previously demonstrated by Mizuno, Warnaar, and B. Wang–L. Wang. Building upon these three families, we construct two vector-valued automorphic forms, one of which is a vector-valued modular function when r is odd.

1. Introduction

A Nahm sum or Nahm series was defined by Nahm [16–18] as the following class of q-hypergeometric series:

$$f_{A,\boldsymbol{b},c}(q) := \sum_{\boldsymbol{n}=(n_1,\ldots,n_r)^T \in \mathbb{N}^r} \frac{q^{\frac{1}{2}\boldsymbol{n}^T A\boldsymbol{n}+\boldsymbol{n}^T\boldsymbol{b}+c}}{(q;q)_{n_1}\cdots(q;q)_{n_r}}, \tag{1.1}$$

where $r \geq 1$ is a positive integer, $A \in \mathbb{Q}^{r\times r}$ is a real positive definite symmetric matrix, $\boldsymbol{b} \in \mathbb{Q}^r$ is a vector and $c \in \mathbb{Q}$ is a scalar. Here and throughout the paper, we adopt the standard notation on q-series [1, 8]:

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

$$(a_1,a_2,\ldots,a_k;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_k;q)_n,$$

and

$$(a_1,a_2,\ldots,a_k;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_k;q)_\infty.$$

Nahm [18] proposed the following famous problem: find all $(A,\boldsymbol{b},c)$ with rational entries such that $f_{A,\boldsymbol{b},c}(q)$ is a modular function, where $q=e^{2\pi i\tau}$ and $\tau \in \mathbb{H} := \{\tau \in \mathbb{C} : \text{Im } \tau > 0\}$. Such a triple $(A,\boldsymbol{b},c)$ is called a modular triple. Recall that a modular function is certain meromorphic function defined on $\mathbb{H}$ satisfying modular transformation and certain holomorphic/meromorphic conditions (see Definition

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4.1). Nahm also [18] made a conjecture which provides a necessary and sufficient condition about the matrix A so that it is the matrix part of some modular triples. The conjecture is formulated in terms of the Bloch group and a system of polynomial equations induced by A, see Zagier [33, p. 43] for more details.

In 2007, Zagier [33] made an important progress towards Nahm's problem. He confirmed Nahm's conjecture for $r = 1$ by showing that there are exactly seven modular triples. For $r \geq 2$, Nahm's conjecture is known to be false in general. Vlasenko and Zwegers [24] provided a counterexample when $r = 2$, that is, for $A = \left(\begin{smallmatrix} 3/4 & -1/4 \\ -1/4 & 3/4 \end{smallmatrix}\right)$, there exists a $\boldsymbol{b}$ and c such that $f_{A,\boldsymbol{b},c}(q)$ is modular, but not all solutions of Nahm's equation give a torsion element in the Bloch group. For $r = 2$ and 3, Zagier [33] found many possible modular triples, which were later affirmed by Cao–Rosengren–Wang [3], Cherednik–Feigin [6], Vlasenko–Zwegers [24], Wang [29, 30] and Zagier [33].

According to physical predictions, Zagier [33] expects that if a Nahm sum associated with a matrix A is modular, then the collection of all modular Nahm sums for A spans a vector space that is invariant under the action of $SL_2(\mathbb{Z})$ in the bosonic case, or at least under $\Gamma(2)$ in the fermionic case. Note that in the language of the physicists, for a sum-product identity, the sum side is called a "fermionic" representation, the product side is called a "bosonic" representation, see [22]. Furthermore, Zagier provided explicit transformation formulas for the vector-valued function arising from the Rogers–Ramanujan identities under the action of $SL_2(\mathbb{Z})$. Building on this, Milas and Wang [14] established analogous transformation laws for vector-valued functions composed of generalized tadpole Nahm sums.

Recently, Mizuno [15] considered generalized Nahm sums associated with symmetrizable matrices, which take the form:

$$\widetilde{f}_{A,\boldsymbol{b},c,\boldsymbol{d}}(q) := \sum_{\boldsymbol{n}=(n_1,\ldots,n_r)^T \in \mathbb{N}^r} \frac{q^{\frac{1}{2}\boldsymbol{n}^T AD\boldsymbol{n}+\boldsymbol{n}^T\boldsymbol{b}+c}}{(q^{d_1};q^{d_1})_{n_1}\cdots(q^{d_r};q^{d_r})_{n_r}}, \tag{1.2}$$

where $\boldsymbol{d} = (d_1, \ldots, d_r) \in \mathbb{Z}^r_{>0}$, $\boldsymbol{b} \in \mathbb{Q}^r$ is a vector and $c \in \mathbb{Q}$ is a scalar. Denote by $D := \mathrm{diag}(d_1, \ldots, d_r)_{r\times r}$ the $r \times r$ diagonal matrix with diagonal entries $d_1, \ldots, d_r$. A matrix $A \in \mathbb{Q}^{r\times r}$ is said to be a symmetrizable matrix with the symmetrizer $D = \mathrm{diag}(d_1, \ldots, d_r)_{r\times r}$ if AD is symmetric positive definite.

Following the notation in [15] and [28], we call $\widetilde{f}_{A,\boldsymbol{b},c,\boldsymbol{d}}(q)$ in (1.2) a Nahm sum for the symmetrizable matrix A of index $(d_1, \ldots, d_r)$. When $\widetilde{f}_{A,\boldsymbol{b},c,\boldsymbol{d}}(q)$ is a modular function, the quadruple $(A, \boldsymbol{b}, c, \boldsymbol{d})$ is called a modular quadruple or such a triple $(A, \boldsymbol{b}, c)$ is called a modular triple of index $(d_1, \ldots, d_r)$.

Generalized Nahm sums of this kind are common in partition identities and the study of affine Lie algebras. For example, one of Capparelli's partition identities [4] asserts that

$$\sum_{n_1,n_2\geq 0}\frac{q^{2n_1^2+6n_1n_2+6n_2^2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}=(-q^2,-q^3,-q^4,-q^6;q^6)_\infty, \tag{1.3}$$

which was first discovered by Capparelli in purely combinatorial form via the theory of affine Lie algebras. The double-sum representation on the left-hand side of (1.3) was discovered by Kanade–Russell [11] and Kurşungöz [13] independently. Notice that (1.3) reveals that the generalized Nahm sum $\widetilde{f}_{A,\boldsymbol{b},c,\boldsymbol{d}}(q)$ associated with $A=\begin{pmatrix}4&2\\6&4\end{pmatrix}$, $\boldsymbol{b}=(0,0)^{\mathrm{T}}$, $c=-1/24$ and $\boldsymbol{d}=(1,3)$ is modular.

Mizuno [15] found that the theory of Nahm sums for symmetric matrices can be analogously applied to Nahm sums for symmetrizable matrices. For example, following the strategy of Zagier [33], Mizuno [15] proposed potential modular triples of indices $(1,2)$, $(1,3)$ and $(1,4)$, and modular triples of indices $(1,1,2)$ and $(2,2,1)$, which have been confirmed by B. Wang and L. Wang [26–28] recently.

Mizuno [15, Eqs. (45), (54)] conjectured two vector-valued transformation formulas involving generalized Nahm sums for symmetrizable matrices. In particular, Mizuno observed that there seem to be "Langlands dual" pairs of modular Nahm sums on the rank 3 case. He conjectured two Nahm sums associated with

$$A=\begin{pmatrix}1&0&1/2\\0&2&1\\1&2&2\end{pmatrix}\quad\text{and}\quad A^\vee=\begin{pmatrix}1&0&1\\0&2&2\\1/2&1&2\end{pmatrix}$$

which are related by taking the transpose, are related by the modular S-transformation. Mizuno's conjecture was subsequently proved by B. Wang–L. Wang [27, 28].

While Nahm's problem has been studied in numerous examples for rank $r\leq 3$, a solution for the general case remains elusive. In this paper, we present three families of modular Nahm sums for symmetrizable matrices with arbitrary rank $r\geq 2$ of indices $(2,\ldots,2,1)$ and $(1,\ldots,1,2)$. Based on these three families of generalized Nahm sums, we construct two vector-valued automorphic forms, one of which is a vector-valued modular function if r is odd (see Section 5 for the relevant definitions).

Theorem 1.1. *For $r\geq 2$ and $0\leq j\leq r$, the Nahm sums $\widetilde{f}_{A,\boldsymbol{b}_j,c_j,\boldsymbol{d}}(q^{32r-8})$ for the symmetrizable matrix A of index $\boldsymbol{d}=(2,\ldots,2,1)$ are modular functions for $\Gamma_1(128(4r-1)^2)$, where*

$$A=\begin{pmatrix}1&0&0&0&\cdots&0&0&1\\0&2&2&2&\cdots&2&2&2\\0&2&4&4&\cdots&4&4&4\\0&2&4&6&\cdots&6&6&6\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\0&2&4&6&\cdots&2(r-3)&2(r-3)&2(r-3)\\0&2&4&6&\cdots&2(r-3)&2(r-2)&2(r-2)\\\frac{1}{2}&1&2&3&\cdots&r-3&r-2&r-1\end{pmatrix}_{r\times r},$$

$$\boldsymbol{b}_0=(0,\ldots,0)^T_{1\times r},\quad c_0=\frac{5-4r}{32r-8},$$

and for $1\leq j\leq r$,

$$\boldsymbol{b}_j=(1,0,\ldots,0,2,4,\ldots,2(r-1-j),r-j)^T_{1\times r},\quad c_j=\frac{(4r-4j-1)^2}{32r-8}.$$

Theorem 1.2. *For* $r\geq 2$ *and* $0\leq j\leq r$*, the Nahm sums* $\widetilde{f}_{B,\boldsymbol{b}_j,c_j,\boldsymbol{d}}(q^{32r-24})$ *for the symmetrizable matrix* B *of index* $\boldsymbol{d}=(2,\ldots,2,1)$ *are modular functions for* $\Gamma_1(64(4r-3)^2)$*, where*

$$B=\begin{pmatrix}1&0&0&0&\cdots&0&0&1\\0&2&2&2&\cdots&2&2&2\\0&2&4&4&\cdots&4&4&4\\0&2&4&6&\cdots&6&6&6\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\0&2&4&6&\cdots&2(r-3)&2(r-3)&2(r-3)\\0&2&4&6&\cdots&2(r-3)&2(r-2)&2(r-2)\\\frac{1}{2}&1&2&3&\cdots&r-3&r-2&r-\frac{1}{2}\end{pmatrix}_{r\times r},$$

$$\boldsymbol{b}_0=\left(0,\ldots,0,-\frac{1}{2}\right)^T_{1\times r},\quad c_0=\frac{7-4r}{32r-24},$$

and for $1\leq j\leq r$,

$$\boldsymbol{b}_j=(1,0,\ldots,0,2,4,\ldots,2(r-1-j),r-j-1)^T_{1\times r},\quad c_j=\frac{(4r-4j-3)^2}{32r-24}.$$

Theorem 1.3. *For* $r\geq 2$ *and* $0\leq j\leq r$*, the Nahm sums* $\widetilde{f}_{A^\vee,\boldsymbol{b}_j,c_j,\boldsymbol{d}^\vee}(q^{32r-8})$ *for the symmetrizable matrix* $A^\vee$ *of index* $\boldsymbol{d}^\vee=(1,\ldots,1,2)$ *are modular functions for*

$\Gamma_1(128(4r-1)^2)$, *where*

$$A^\vee = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2} \\ 0 & 2 & 2 & 2 & \cdots & 2 & 2 & 1 \\ 0 & 2 & 4 & 4 & \cdots & 4 & 4 & 2 \\ 0 & 2 & 4 & 6 & \cdots & 6 & 6 & 3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-3) & r-3 \\ 0 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-2) & r-2 \\ 1 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-2) & r-1 \end{pmatrix}_{r\times r},$$

$$\boldsymbol{b}_0 = (1,1,2,3,\ldots,r-1)^T_{1\times r}, \quad c_0 = \frac{8r^2-14r+5}{32r-8},$$

and for $1 \le j \le r$,

$$\boldsymbol{b}_j = (0,\ldots,0,1,2,\ldots,r-j)^T_{1\times r}, \quad c_j = \frac{8(r-j)^2+4j-6r+1}{32r-8}.$$

We now provide several remarks on Theorems 1.1–1.3.

Remark 1.4. (1) *The symmetrizable matrix A in Theorem 1.1 is the transpose of the symmetrizable matrix $A^\vee$ in Theorem 1.3.*

(2) *The cases $r = 2$ and $r = 3$ in Theorem 1.1 and Theorem 1.2 were conjectured by Mizuno (see [15, Table 1, Table 2, Table 3]). The case $r = 2$ in Theorem 1.1 was confirmed by B. Wang–L. Wang [28]. For $r = 2$ in Theorem 1.2, the case b_0 was established by Warnaar [32, (5.14)], as observed by Mizuno, while the remaining cases were proved by Mizuno [15]. The cases $r = 3$ in both Theorem 1.1 and Theorem 1.2 were confirmed by B. Wang–L. Wang [27].*

(3) *The matrices A and B in Theorem 1.1 and Theorem 1.2 are symmetrizable matrices with the symmetrizer $D = \mathrm{diag}(2,\ldots,2,1)_{r\times r}$ since the positive definiteness of the symmetric matrices AD and BD given in* (1.4) *and* (1.5) *are easily verified.*

$$AD = \begin{pmatrix} 2 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 4 & 4 & 4 & \cdots & 4 & 4 & 2 \\ 0 & 4 & 8 & 8 & \cdots & 8 & 8 & 4 \\ 0 & 4 & 8 & 12 & \cdots & 12 & 12 & 6 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 4 & 8 & 12 & \cdots & 4(r-3) & 4(r-3) & 2(r-3) \\ 0 & 4 & 8 & 12 & \cdots & 4(r-3) & 4(r-2) & 2(r-2) \\ 1 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-2) & r-1 \end{pmatrix}_{r\times r}, \tag{1.4}$$

$$BD = \begin{pmatrix} 2 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 4 & 4 & 4 & \cdots & 4 & 4 & 2 \\ 0 & 4 & 8 & 8 & \cdots & 8 & 8 & 4 \\ 0 & 4 & 8 & 12 & \cdots & 12 & 12 & 6 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 4 & 8 & 12 & \cdots & 4(r-3) & 4(r-3) & 2(r-3) \\ 0 & 4 & 8 & 12 & \cdots & 4(r-3) & 4(r-2) & 2(r-2) \\ 1 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-2) & r-\frac{1}{2} \end{pmatrix}_{r\times r}. \tag{1.5}$$

(4) *Similarly, the matrix* $A^\vee$ *in Theorem 1.3 is a symmetrizable matrix with the symmetrizer* $D^\vee = \mathrm{diag}(1,\dots,1,2)_{r\times r}$ *since* $A^\vee D^\vee$ *as shown in* (1.6) *is a symmetric positive definite matrix.*

$$A^\vee D^\vee = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 2 & 2 & 2 & \cdots & 2 & 2 & 2 \\ 0 & 2 & 4 & 4 & \cdots & 4 & 4 & 4 \\ 0 & 2 & 4 & 6 & \cdots & 6 & 6 & 6 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-3) & 2(r-3) \\ 0 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-2) & 2(r-2) \\ 1 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-2) & 2(r-1) \end{pmatrix}_{r\times r}. \tag{1.6}$$

To prove Theorems 1.1, 1.2 and 1.3, we establish the Rogers–Ramanujan type identities stated in Theorem 1.5. Each multi-sum appearing in Theorem 1.5 corresponds to the generalized Nahm sums $\widetilde{f}_{A,\boldsymbol{b},c,\boldsymbol{d}}(q)$, specified in Theorems 1.1, 1.2 and 1.3, whereas each product term can be expressed as a generalized eta-quotient or a sum of generalized eta-quotients, whose modularity can be determined using the criterion established by Robins [19, Theorem 3]. It should be noted that the identities (1.11) and (1.12) in Theorem 1.5 are equivalent to those due to B. Wang–L. Wang [26, Corollary 4.2 and Corollary 4.3] and we include them in Theorem 1.5 for completeness.

Theorem 1.5. *For* $r \geq 2$,

(1) $$\sum_{\boldsymbol{n}=(n_1,\dots,n_r)^T\in\mathbb{N}^r} \frac{q^{\frac{1}{2}\boldsymbol{n}^T AD\boldsymbol{n}+\boldsymbol{n}^T\boldsymbol{b}_0}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{n_r}}$$

$$= \frac{(q^{8r-4},q^{8r},q^{16r-4};q^{16r-4})_\infty}{(q,q^3,q^4;q^4)_\infty} + q^{\frac{r-1}{2}}\frac{(-q,q^{4r-2},-q^{4r-1};-q^{4r-1})_\infty}{(q^2,q^2,q^4;q^4)_\infty}, \tag{1.7}$$

where AD *is given by* (1.4) *and* $\boldsymbol{b}_0 = (0,0,\cdots,0,0)^T_{1\times r}$.

(2) *For* $1\le j\le r$,

$$\sum_{\boldsymbol{n}=(n_1,\dots,n_r)^T\in\mathbb{N}^r}\frac{q^{\frac{1}{2}\boldsymbol{n}^TAD\boldsymbol{n}+\boldsymbol{n}^T\boldsymbol{b}_j}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{n_r}} \tag{1.8}$$
$$=\frac{(q^{2j},-q^{4r-2j-1},-q^{4r-1};-q^{4r-1})_\infty}{(q^2,q^2,q^4;q^4)_\infty}+q^{\frac{3r-2j-1}{2}}\frac{(q^{4j},q^{16r-4j-4},q^{16r-4};q^{16r-4})_\infty}{(q,q^3,q^4;q^4)_\infty},$$

where AD *is given by* (1.4) *and* $\boldsymbol{b}_j=(1,0,\cdots,0,2,4,\cdots,2(r-1-j),r-j)^T_{1\times r}$.

(3)
$$\sum_{\boldsymbol{n}=(n_1,\dots,n_r)^T\in\mathbb{N}^r}\frac{q^{\frac{1}{2}\boldsymbol{n}^TBD\boldsymbol{n}+\boldsymbol{n}^T\boldsymbol{b}_0}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{n_r}}$$
$$=\frac{(q^{8r-8},q^{8r-4},q^{16r-12};q^{16r-12})_\infty}{(q,q^3,q^4;q^4)_\infty}+q^{\frac{2r-3}{4}}\frac{(-q,q^{4r-4},-q^{4r-3};-q^{4r-3})_\infty}{(q^2,q^2,q^4;q^4)_\infty}, \tag{1.9}$$

where BD *is given by* (1.5) *and* $\boldsymbol{b}_0=(0,0,\cdots,0,-\frac{1}{2})^T_{1\times r}$.

(4) *For* $1\le j\le r$,

$$\sum_{\boldsymbol{n}=(n_1,\dots,n_r)\in\mathbb{N}^r}\frac{q^{\frac{1}{2}\boldsymbol{n}^TBD\boldsymbol{n}+\boldsymbol{n}^T\boldsymbol{b}_j}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{n_r}} \tag{1.10}$$
$$=\frac{(q^{2j},-q^{4r-3-2j},-q^{4r-3};-q^{4r-3})_\infty}{(q^2,q^2,q^4;q^4)_\infty}+q^{\frac{6r-4j-5}{4}}\frac{(q^{4j},q^{16r-4j-12},q^{16r-12};q^{16r-12})_\infty}{(q,q^3,q^4;q^4)_\infty},$$

where BD *is given by* (1.5) *and* $\boldsymbol{b}_j=(1,0,\cdots,0,2,4,\cdots,2(r-j-1),r-j-1)^T_{1\times r}$.

(5)
$$\sum_{\boldsymbol{n}=(n_1,\dots,n_r)^T\in\mathbb{N}^r}\frac{q^{\frac{1}{2}\boldsymbol{n}^TA^\vee D^\vee\boldsymbol{n}+\boldsymbol{n}^T\boldsymbol{b}_0}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{r-1}}(q^2;q^2)_{n_r}}=\frac{(q^{\frac{1}{2}},q^{2r-1},q^{2r-\frac{1}{2}};q^{2r-\frac{1}{2}})_\infty}{(q^{\frac{1}{2}},q^{\frac{3}{2}},q^2;q^2)_\infty}, \tag{1.11}$$

where $A^\vee D^\vee$ *is given by* (1.6) *and* $\boldsymbol{b}_0=(1,1,2,3,\dots,r-1)^T_{1\times r}$.

(6) *For* $1\le j\le r$,

$$\sum_{\boldsymbol{n}=(n_1,\dots,n_r)^T\in\mathbb{N}^r}\frac{q^{\frac{1}{2}\boldsymbol{n}^TA^\vee D^\vee\boldsymbol{n}+\boldsymbol{n}^T\boldsymbol{b}_j}}{(q;q)_{n_1}\cdots(q;q)_{n_{r-1}}(q^2;q^2)_{n_r}}=\frac{(q^j,q^{2r-j-\frac{1}{2}},q^{2r-\frac{1}{2}};q^{2r-\frac{1}{2}})_\infty}{(q^{\frac{1}{2}},q^{\frac{3}{2}},q^2;q^2)_\infty}, \tag{1.12}$$

where $A^\vee D^\vee$ *is given by* (1.6) *and* $\boldsymbol{b}_j = (0, \ldots, 0, 1, 2, \ldots, r-j)^T_{1\times r}$.

Based on these three families of generalized Nahm sums, we introduce the following two vector-valued functions $\boldsymbol{G}_{4r-2}(\tau)$ and $\boldsymbol{H}_{4r-4}(\tau)$:

$$\boldsymbol{G}_{4r-2}(\tau) = (g_1(\tau), g_2(\tau), \ldots, g_{2r-1}(\tau), g_1^\vee(\tau), g_2^\vee(\tau), \ldots, g_{2r-1}^\vee(\tau))^T$$

and

$$\boldsymbol{H}_{4r-4}(\tau) = (h_1(\tau), h_2(\tau), \ldots, h_{2r-2}(\tau), h_1^\vee(\tau), h_2^\vee(\tau), \ldots, h_{2r-2}^\vee(\tau))^T,$$

where

$$g_j(\tau) = q^{\frac{(4r-4j+1)^2}{32r-8}} \left(q^{\frac{2j-r-1}{2}} \frac{(q^{8r-4j}, q^{8r+4j-4}, q^{16r-4}; q^{16r-4})_\infty}{(q, q^3, q^4; q^4)_\infty} + \frac{(-q^{2j-1}, q^{4r-2j}, -q^{4r-1}; -q^{4r-1})_\infty}{(q^2, q^2, q^4; q^4)_\infty} \right), \tag{1.13}$$

$$g_j^\vee(2\tau) = \frac{1}{2} q^{\frac{2j^2-2j-2r+1}{16r-4}} \left(\frac{(q^{2r-j}, q^{2r+j-1}, q^{4r-1}; q^{4r-1})_\infty}{(q, q^3, q^4; q^4)_\infty} + (-1)^{\frac{j(j-1)}{2}} \frac{((-q)^{2r-j}, (-q)^{2r+j-1}, -q^{4r-1}; -q^{4r-1})_\infty}{(-q, -q^3, q^4; q^4)_\infty} \right), \tag{1.14}$$

$$h_j(\tau) = q^{\frac{(4j-4r+1)^2}{32r-24}} \left(q^{\frac{4j-2r-1}{4}} \frac{(q^{4(2r-1-j)}, q^{8r+4j-8}, q^{16r-12}; q^{16r-12})_\infty}{(q, q^3, q^4; q^4)_\infty} + \frac{(q^{2(2r-j-1)}, -q^{2j-1}, -q^{4r-3}; -q^{4r-3})_\infty}{(q^2, q^2, q^4; q^4)_\infty} \right), \tag{1.15}$$

$$h_j^\vee(2\tau) = \frac{1}{2} q^{\frac{j^2-j-r+1}{8r-6}} \left(\frac{(q^{2r-j-1}, q^{2r+j-2}, q^{4r-3}; q^{4r-3})_\infty}{(q, q^3, q^4; q^4)_\infty} + \frac{((-q)^{2r-j-1}, (-q)^{2r+j-2}, -q^{4r-3}; -q^{4r-3})_\infty}{(-q, -q^3, q^4; q^4)_\infty} \right). \tag{1.16}$$

We then establish the following results.

Theorem 1.6. *For* $r \geq 2$, $\boldsymbol{G}_{4r-2}(\tau)$ *is a vector-valued automorphic form of some multiplier for the group generated by* $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)$. *In particular, for odd integer* $r \geq 3$, $\boldsymbol{G}_{4r-2}(\tau)$ *is a vector-valued modular function of some multiplier with respect to the congruence subgroup* $\Gamma_0(2)$.

Theorem 1.7. *For* $r \geq 2$, $\boldsymbol{H}_{4r-4}(\tau)$ *is a vector-valued automorphic form of some multiplier for the group generated by* $\left(\begin{smallmatrix} 1 & 4 \\ 0 & 1 \end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix} 1 & 0 \\ 8 & 1 \end{smallmatrix}\right)$.

Several remarks are in order.

Remark 1.8. (1) *In order to prove Theorem 1.6 and Theorem 1.7, we establish modular transformation formulas on "Langlands dual" pair* $\boldsymbol{g}_{2r-1}(\tau)$ *and* $\boldsymbol{g}^\vee_{2r-1}(\tau)$*, and* $\boldsymbol{h}_{2r-2}(\tau)$ *and* $\boldsymbol{h}^\vee_{2r-2}(\tau)$ *(see Theorem 5.5 and Theorem 5.7), where*

$$\boldsymbol{g}_{2r-1}(\tau) = (g_1(\tau), g_2(\tau), \ldots, g_{2r-1}(\tau))^T, \quad \boldsymbol{g}^\vee_{2r-1}(\tau) = (g^\vee_1(\tau), g^\vee_2(\tau), \ldots, g^\vee_{2r-1}(\tau))^T,$$

and

$$\boldsymbol{h}_{2r-2}(\tau) = (h_1(\tau), h_2(\tau), \ldots, h_{2r-2}(\tau))^T, \quad \boldsymbol{h}^\vee_{2r-2}(\tau) = (h^\vee_1(\tau), h^\vee_2(\tau), \ldots, h^\vee_{2r-2}(\tau))^T.$$

Notice that the case $r = 3$ *of Theorem 5.5 was conjectured by Mizuno [15, Eq. (54)]), and subsequently confirmed by B. Wang–L. Wang [27].*

(2) *It follows from Theorem 1.5 (1) and (2) that*

$$g_1(\tau) = \widetilde{f}_{A,\boldsymbol{b}_0,c_0,\boldsymbol{d}}(\tau),$$

and, for $r \le j \le 2r-1$*, we have*

$$g_j(\tau) = \widetilde{f}_{A,\boldsymbol{b}_{2r-j},c_{2r-j},\boldsymbol{d}}(\tau),$$

where $(A, \boldsymbol{b}_j, c_j)$ *is the triple of index* $\boldsymbol{d} = (2, \ldots, 2, 1)$ *given in Theorem 1.1.*

Moreover, by Theorem 1.5 (5) and (6), we have

$$g^\vee_{2r-1}(2\tau) = \frac{1}{2}\left(\widetilde{f}_{A^\vee,2\boldsymbol{b}_0,2c_0,2\boldsymbol{d}^\vee}(\tau) + (-1)^{r-1}\widetilde{f}_{A^\vee,2\boldsymbol{b}_0,2c_0,2\boldsymbol{d}^\vee}\Big(\tau + \frac{1}{2}\Big)\right),$$

and, for $2 \le j \le 2r-2$ *and* j *is even, we have*

$$g^\vee_j(2\tau) = \frac{1}{2}\left(\widetilde{f}_{A^\vee,2\boldsymbol{b}_{r-\frac{j}{2}},2c_{r-\frac{j}{2}},2\boldsymbol{d}^\vee}(\tau) + (-1)^{\frac{j(j-1)}{2}}\widetilde{f}_{A^\vee,2\boldsymbol{b}_{r-\frac{j}{2}},2c_{r-\frac{j}{2}},2\boldsymbol{d}^\vee}\Big(\tau + \frac{1}{2}\Big)\right),$$

where $(A^\vee, \boldsymbol{b}_i, c_i)$ *is the triple of index* $\boldsymbol{d}^\vee = (1, \ldots, 1, 2)$ *given in Theorem 1.3.*

(3) *By Theorem 1.5 (3) and (4), we obtain*

$$h_1(\tau) = \widetilde{f}_{B,\boldsymbol{b}_0,c_0,\boldsymbol{d}}(\tau),$$

and for $r-1 \le j \le 2r-2$,

$$h_j(\tau) = \widetilde{f}_{B,\boldsymbol{b}_{2r-j-1},c_{2r-j-1},\boldsymbol{d}}(\tau),$$

where $(B, \boldsymbol{b}_j, c_j)$ *is the triple of index* $\boldsymbol{d} = (2, \ldots, 2, 1)$ *given in Theorem 1.2.*

When $r \ge 3$ and $j = 2$, we find that $g_2(\tau)$ can be expressed as the sum of two Nahm sums for the symmetrizable matrix A of index $\boldsymbol{d} = (2, \ldots, 2, 1)$, as shown in the following proposition.

Proposition 1.9. *We have*

$$g_2(\tau) = \widetilde{f}_{A,\boldsymbol{b}_{r+1},c_{r+1},\boldsymbol{d}}(\tau) + \widetilde{f}_{A,\widetilde{\boldsymbol{b}}_{r+1},\widetilde{c}_{r+1},\boldsymbol{d}}(\tau),$$

where A is the symmetrizable matrix with the symmetrizer $D = \mathrm{diag}(2, \dots, 2, 1)_{r\times r}$ given in Theorem 1.1,

$$\boldsymbol{b}_{r+1} = (0, \dots, 0, 1)^T_{1\times r}, \quad c_{r+1} = \frac{37-4r}{32r-8},$$
$$\widetilde{\boldsymbol{b}}_{r+1} = (1, 2, 4, \dots, 2(r-2), r)^T_{1\times r}, \quad \widetilde{c}_{r+1} = \frac{(4r-7)^2}{32r-8}.$$

This relation is implied by the following identity.

Theorem 1.10. *For $r \geq 3$,*

$$\sum_{\boldsymbol{n}=(n_1,\dots,n_r)^T\in\mathbb{N}^r} \frac{q^{\frac{1}{2}\boldsymbol{n}^TAD\boldsymbol{n}+\boldsymbol{n}^T\boldsymbol{b}_{r+1}} + q^{\frac{1}{2}\boldsymbol{n}^TAD\boldsymbol{n}+\boldsymbol{n}^T\widetilde{\boldsymbol{b}}_{r+1}+\frac{r-3}{2}}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{n_r}}$$
$$= \frac{(q^{8r+4}, q^{8r-8}, q^{16r-4}; q^{16r-4})_\infty}{(q, q^3, q^4; q^4)_\infty} + q^{\frac{r-3}{2}} \frac{(-q^3, q^{4r-4}, -q^{4r-1}; -q^{4r-1})_\infty}{(q^2, q^2, q^4; q^4)_\infty}, \tag{1.17}$$

where AD is given by (1.4), *$\boldsymbol{b}_{r+1} = (0, \dots, 0, 1)^T_{1\times r}$, and $\widetilde{\boldsymbol{b}}_{r+1} = (1, 2, 4, \dots, 2(r-2), r)^T_{1\times r}$.*

Notice that the case $r = 3$ of Theorem 1.10 has been given by B. Wang–L. Wang [27, Eq. (1.13)].

To express $h_j^\vee(2\tau)$ as a combination of generalized Nahm sums of index $(1, \dots, 1, 2)$, we employ the notion of partial Nahm sums introduced in [31]. For the Nahm sum associated with symmetrizable matrix A:

$$\widetilde{f}_{A,\boldsymbol{b},c,\boldsymbol{d}}(q) := \sum_{\boldsymbol{n}=(n_1,\dots,n_r)^T\in\mathbb{N}^r} \frac{q^{\frac{1}{2}\boldsymbol{n}^TAD\boldsymbol{n}+\boldsymbol{n}^T\boldsymbol{b}+c}}{(q^{d_1};q^{d_1})_{n_1}\cdots(q^{d_r};q^{d_r})_{n_r}},$$

its partial Nahm sum is defined as

$$\widetilde{f}_{A,\boldsymbol{b},c,\boldsymbol{d},\sigma}(q) := \sum_{\substack{\boldsymbol{n}=(n_1,\dots,n_r)\in\mathbb{N}^r \\ n_r\equiv\sigma \pmod 2}} \frac{q^{\frac{1}{2}\boldsymbol{n}^TAD\boldsymbol{n}+\boldsymbol{n}^T\boldsymbol{b}+c}}{(q^{d_1};q^{d_1})_{n_1}\cdots(q^{d_r};q^{d_r})_{n_r}},$$

where $\sigma = 0, 1$.

It turns out that $h_j^\vee(2\tau)$ can be expressed as a combination of the generalized Nahm sums $\widetilde{f}_{B^\vee,\boldsymbol{b}_j,c_j,\boldsymbol{d}^\vee}(q)$ of index $\boldsymbol{d}^\vee = (1, \dots, 1, 2)$, where $(B^\vee, \boldsymbol{b}_j, c_j)$ are given by

$$B^\vee = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2} \\ 0 & 2 & 2 & 2 & \cdots & 2 & 2 & 1 \\ 0 & 2 & 4 & 4 & \cdots & 4 & 4 & 2 \\ 0 & 2 & 4 & 6 & \cdots & 6 & 6 & 3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-3) & r-3 \\ 0 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-2) & r-2 \\ 1 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-2) & r-\frac{1}{2} \end{pmatrix}_{r\times r}, \tag{1.18}$$

$$\boldsymbol{b}_0 = (1,1,2,3,\dots,r-1)^T_{1\times r}, \quad c_0 = \frac{4r^2-11r+7}{16r-12}, \tag{1.19}$$

and for $1 \le j \le r$,

$$\boldsymbol{b}_j = (0,\dots,0,1,2,\dots,r-j-1,r-j-1)^T_{1\times r}, \quad c_j = \frac{4(r-j)^2+6j-7r+3}{16r-12}. \tag{1.20}$$

It is straightforward to verify that $B^\vee D^\vee$ as shown in (1.21) is a symmetric positive definite matrix.

$$B^\vee D^\vee = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 2 & 2 & 2 & \cdots & 2 & 2 & 2 \\ 0 & 2 & 4 & 4 & \cdots & 4 & 4 & 4 \\ 0 & 2 & 4 & 6 & \cdots & 6 & 6 & 6 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-3) & 2(r-3) \\ 0 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-2) & 2(r-2) \\ 1 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-2) & 2r-1 \end{pmatrix}_{r\times r}. \tag{1.21}$$

Note that the symmetrizable matrix $B^\vee$ given in (1.18) is the transpose of the symmetrizable matrix B in Theorem 1.2.

Proposition 1.11. *We have*

$$h^\vee_{2r-2}(2\tau) = \frac{1}{2}\Big(\widetilde{f}_{B^\vee,2\boldsymbol{b}_0,2c_0,2\boldsymbol{d}^\vee,0}(\tau) - \widetilde{f}_{B^\vee,2\boldsymbol{b}_0,2c_0,2\boldsymbol{d}^\vee,1}(\tau) + \widetilde{f}_{B^\vee,2\boldsymbol{b}_0,2c_0,2\boldsymbol{d}^\vee,0}\Big(\tau+\frac{1}{2}\Big) - \widetilde{f}_{B^\vee,2\boldsymbol{b}_0,2c_0,2\boldsymbol{d}^\vee,1}\Big(\tau+\frac{1}{2}\Big)\Big),$$

and for $1 \le j \le 2r-3$ and j is odd, we have

$$h^\vee_j(2\tau) = \frac{1}{2}\Big(\widetilde{f}_{B^\vee,2\boldsymbol{b}_{r-\frac{j+1}{2}},2c_{r-\frac{j+1}{2}},2\boldsymbol{d}^\vee,0}(\tau) - \widetilde{f}_{B^\vee,2\boldsymbol{b}_{r-\frac{j+1}{2}},2c_{r-\frac{j+1}{2}},2\boldsymbol{d}^\vee,1}(\tau)$$

$$+\widetilde{f}_{B^\vee,2\boldsymbol{b}_{r-\frac{j+1}{2}},2c_{r-\frac{j+1}{2}},2\boldsymbol{d}^\vee,0}\Big(\tau+\frac{1}{2}\Big)-\widetilde{f}_{B^\vee,2\boldsymbol{b}_{r-\frac{j+1}{2}},2c_{r-\frac{j+1}{2}},2\boldsymbol{d}^\vee,1}\Big(\tau+\frac{1}{2}\Big)\Big).$$

This relation follows directly from the following result:

Theorem 1.12. *We have*

(1)
$$\sum_{\boldsymbol{n}=(n_1,\ldots,n_r)^T\in\mathbb{N}^r}\frac{(-1)^{n_r}q^{\frac{1}{2}\boldsymbol{n}^TB^\vee D^\vee\boldsymbol{n}+\boldsymbol{n}^T\boldsymbol{b}_0}}{(q;q)_{n_1}\cdots(q;q)_{n_{r-1}}(q^2;q^2)_{n_r}}=\frac{(q^{\frac{1}{2}},q^{2r-2},q^{2r-\frac{3}{2}};q^{2r-\frac{3}{2}})_\infty}{(q^{\frac{1}{2}},q^{\frac{3}{2}},q^2;q^2)_\infty},\tag{1.22}$$

where $B^\vee D^\vee$ *is given by* (1.21) *and* $\boldsymbol{b}_0=(1,1,2,3,\ldots,r-1)^T_{1\times r}$,

(2)*for* $1\le j\le r$,
$$\sum_{\boldsymbol{n}=(n_1,\ldots,n_r)^T\in\mathbb{N}^r}\frac{(-1)^{n_r}q^{\frac{1}{2}\boldsymbol{n}^TB^\vee D^\vee\boldsymbol{n}+\boldsymbol{n}^T\boldsymbol{b}_j}}{(q;q)_{n_1}\cdots(q;q)_{n_{r-1}}(q^2;q^2)_{n_r}}=\frac{(q^{j},q^{2r-j-\frac{3}{2}},q^{2r-\frac{3}{2}};q^{2r-\frac{3}{2}})_\infty}{(q^{\frac{1}{2}},q^{\frac{3}{2}},q^2;q^2)_\infty},\tag{1.23}$$

where $B^\vee D^\vee$ *is given by* (1.21) *and* $\boldsymbol{b}_j=(0,\ldots,0,1,2,\ldots,r-j-1,r-j-1)^T_{1\times r}$.

It remains to address the following problem:

Problem 1.13. (*a*) *For* $r\ge 4$ *and* $3\le j\le r-1$*, can* $g_j(\tau)$ *be expressed as a combination of Nahm sums for the symmetrizable matrix* A *given in Theorem 1.1 of index* $(2,\ldots,2,1)$*?*

(*b*) *For* $r\ge 2$ *and* $1\le j\le 2r-3$ *and* j *is odd, can* $g_j^\vee(2\tau)$ *be expressed as a combination of Nahm sums for the symmetrizable matrix* $A^\vee$ *given in Theorem 1.3 of index* $(1,\ldots,1,2)$*?*

(*c*) *For* $r\ge 4$ *and* $2\le j\le r-2$*, can* $h_j(\tau)$ *be expressed as a combination of Nahm sums for the symmetrizable matrix* B *given in Theorem 1.2 of index* $(2,\ldots,2,1)$*?*

(*d*) *For* $r\ge 3$ *and* $2\le j\le 2r-4$ *and* j *is even, can* $h_j^\vee(2\tau)$ *be expressed as a combination of Nahm sums for the symmetrizable matrix* $B^\vee$ *given in* (1.18) *of index* $(1,\ldots,1,2)$*?*

The paper is organized as follows. In Section 2, we review some relevant known results on the Bailey machinery, and then utilize this machinery to provide proofs of three families of multi-sum Rogers–Ramanujan type identities (see Theorem 2.1, Theorem 2.2 and Theorem 2.3). Section 3 is devoted to proving Theorem 1.5, Theorem 1.10 and Theorem 1.12, using the results of Theorems 2.1, 2.2 and 2.3 in Section 2. In Section 4, we show that the generalized Nahm sums specified in Theorems 1.1, 1.2, 1.3 are modular functions. Section 5 is devoted to proving Theorem 1.6 and Theorem 1.7 by establishing the modular transformation formulas on two "Langlands dual" pairs.

2. Multi-sum Rogers–Ramanujan type identities

In this section, we aim to present the following four families of multi-sum Rogers–Ramanujan type identities which serve as key ingredients in the proof of Theorem 1.5, Theorem 1.10 and Theorem 1.12.

Theorem 2.1. *For $r \geq 2$,*

(*a*)

$$\sum_{N_1 \geq \cdots \geq N_{r-1} \geq 0} \frac{q^{N_1^2+\cdots+N_{r-1}^2}}{(q;q)_{N_1-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}(q;q)_{N_{r-1}}(q;q^2)_{N_{r-1}}} = \frac{(q^{4r-2},q^{4r},q^{8r-2};q^{8r-2})_\infty}{(q;q)_\infty}, \tag{2.1}$$

(*b*) *for $1 \leq j \leq r$,*

$$\sum_{N_1 \geq \cdots \geq N_{r-1} \geq 0} \frac{q^{N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{j-1}+2N_j+\cdots+2N_{r-1}}}{(q;q)_{N_1-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}(q;q)_{N_{r-1}}(q^3;q^2)_{N_{r-1}}} = \frac{(q^{2j},q^{8r-2j-2},q^{8r-2};q^{8r-2})_\infty}{(q^2;q)_\infty}, \tag{2.2}$$

(*c*)

$$\sum_{N_1 \geq \cdots \geq N_{r-1} \geq 0} \frac{q^{N_1^2+\cdots+N_{r-1}^2+\binom{N_{r-1}}{2}}}{(q;q)_{N_1-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}(q;q)_{N_{r-1}}(q;q^2)_{N_{r-1}}} = \frac{(q^{4r-4},q^{4r-2},q^{8r-6};q^{8r-6})_\infty}{(q;q)_\infty}, \tag{2.3}$$

(*d*) *for $1 \leq j \leq r$,*

$$\sum_{N_1 \geq \cdots \geq N_{r-1} \geq 0} \frac{q^{N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{j-1}+2N_j+\cdots+2N_{r-1}+\binom{N_{r-1}}{2}}}{(q;q)_{N_1-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}(q;q)_{N_{r-1}}(q^3;q^2)_{N_{r-1}}} = \frac{(q^{2j},q^{8r-2j-6},q^{8r-6};q^{8r-6})_\infty}{(q^2;q)_\infty}. \tag{2.4}$$

Theorem 2.2. *For $r \geq 2$,*

(*a*)

$$\sum_{N_1 \geq \cdots \geq N_{r-1} \geq 0} \frac{(-1)^{N_{r-1}}q^{2(N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{r-1})+N_{r-1}^2}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^4;q^4)_{N_{r-1}}(-q^3;q^2)_{N_{r-1}}} = \frac{(q,q^{4r-4},q^{4r-3};q^{4r-3})_\infty}{(1-q)(q^4;q^2)_\infty}, \tag{2.5}$$

(*b*) *for* $1 \leq j \leq r$,

$$\sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{(-1)^{N_{r-1}} q^{2(N_1^2+\cdots+N_{r-1}^2+N_j+\cdots+N_{r-1})+N_{r-1}^2-2N_{r-1}}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^4;q^4)_{N_{r-1}}(-q;q^2)_{N_{r-1}}}$$

$$= \frac{(q^{2j},q^{4r-2j-3},q^{4r-3};q^{4r-3})_\infty}{(q^2;q^2)_\infty}. \tag{2.6}$$

Theorem 2.3. *For* $r \geq 3$,

(*a*)
$$\sum_{N_1\geq N_2\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{N_1^2+N_2^2+N_3^2+\cdots+N_{r-1}^2+N_{r-1}}(1+q^{-1}-q^{N_{r-1}-1})}{(q;q)_{N_1-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}(q;q)_{N_{r-1}}(q;q^2)_{N_{r-1}}}$$

$$= \frac{(q^{4r+2},q^{4r-4},q^{8r-2};q^{8r-2})_\infty}{(q;q)_\infty}, \tag{2.7}$$

(*b*)
$$\sum_{N_1\geq N_2\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{N_1^2+N_2^2+N_3^2+\cdots+N_{r-1}^2+N_1+N_2+\cdots N_{r-2}+2N_{r-1}}(1+q+q^{N_{r-1}+\frac{1}{2}})}{(q;q)_{n-N_2}(q;q)_{N_2-N_3}\cdots(q;q)_{N_{r-2}-N_{r-1}}(-q^{1/2};q)_{N_{r-1}+1}(q^2;q^2)_{N_{r-1}}}$$

$$= \frac{(q^{\frac{3}{2}},q^{2r-2},q^{2r-\frac{1}{2}};q^{2r-\frac{1}{2}})_\infty}{(q;q)_\infty}. \tag{2.8}$$

The proofs of Theorems 2.1, 2.2, 2.3 rely on the Bailey machinery. To this end, we review some necessary background on the Bailey machinery in Subsection 2.1. We proceed to lay out the derivations of each identity in Theorem 2.1, Theorem 2.2 and Thereom 2.3 in Subsection 2.2, Subsection 2.3 and Subsection 2.4, respectively.

2.1. **Preliminaries.** We first recall the definition of Bailey pair. A pair of sequences $(\alpha_n(a;q),\beta_n(a;q))$ is called a Bailey pair relative to a if for all $n \geq 0$,

$$\beta_n(a;q) = \sum_{r=0}^{n} \frac{\alpha_r(a;q)}{(q;q)_{n-r}(aq;q)_{n+r}}. \tag{2.9}$$

Note that taking the limit as $n \to \infty$ yields the following expression, subject to appropriate convergence conditions:

$$\lim_{n\to\infty} \beta_n(a;q) = \frac{1}{(q;q)_\infty(aq;q)_\infty} \sum_{r=0}^{\infty} \alpha_r(a;q). \tag{2.10}$$

In practice, the above sum can often be rewritten as an infinite product via an application of the Jacobi triple product identity [1, Theorem 2.8],

$$(q^2,zq,q/z;q^2)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} z^n. \tag{2.11}$$

Once we have a Bailey pair, the following Bailey's lemma provides a method to generate a new Bailey pair.

Lemma 2.4 (Bailey's Lemma)**.** *Suppose that* $(\alpha_n(a;q),\beta_n(a;q))$ *is a Bailey pair relative to* a*. Then* $(\alpha'_n(a;q),\beta'_n(a;q))$ *is also a Bailey pair relative to* a*, where*

$$\begin{aligned}\alpha'_n(a;q)&=\frac{(\rho_1,\rho_2;q)_n(aq/\rho_1\rho_2)^n}{(aq/\rho_1,aq/\rho_2;q)_n}\alpha_n(a;q),\\ \beta'_n(a;q)&=\sum_{r=0}^{n}\frac{(\rho_1,\rho_2;q)_r(aq/\rho_1\rho_2;q)_{n-r}(aq/\rho_1\rho_2)^r}{(aq/\rho_1,aq/\rho_2;q)_n(q;q)_{n-r}}\beta_r(a;q).\end{aligned}\tag{2.12}$$

Upon taking $\rho_1\to\infty,\rho_2\to\infty$, we obtain

Lemma 2.5. [2, Eq. (S1)] *If* $(\alpha_n(a;q),\beta_n(a;q))$ *is a Bailey pair relative to* a*, then* $(\alpha'_n(a;q),\beta'_n(a;q))$ *is also a Bailey pair relative to* a*, where*

$$\alpha'_n(a;q)=a^nq^{n^2}\alpha_n(a;q),\quad \beta'_n(a;q)=\sum_{r=0}^{n}\frac{a^rq^{r^2}}{(q;q)_{n-r}}\beta_r(a;q).\tag{2.13}$$

The following Lemma provides a method for generating a Bailey pair related to a/q from a Bailey pair relative to a.

Lemma 2.6. [10, Lemma 3.1] *If* $(\alpha_n(a;q),\beta_n(a;q))$ *is a Bailey pair relative to* a*, then* $(\alpha'_n(a/q;q),\beta'_n(a/q;q))$ *is a Bailey pair relative to* a/q*, where*

$$\begin{aligned}\alpha'_n(a/q;q)&=(1-a)\left(\frac{q^n\alpha_n(a;q)}{1-aq^{2n}}-\frac{q^{n-1}\alpha_{n-1}(a;q)}{1-aq^{2n-2}}\right),\\ \beta'_n(a/q;q)&=q^n\beta_n(a;q).\end{aligned}\tag{2.14}$$

2.2. **Proof of Theorem 2.1.** We begin by recalling the following Bailey pairs from Group C of Slater's list [23, p. 469], which are required in the proof of Theorem 2.1.

In what follows, we set $\alpha_0(a;q)=1$ and adopt the convention that $\alpha_n(a;q)=0$ for $n<0$ unless specified otherwise.

(C1) $\alpha_{2n}(1;q)=(-1)^nq^{3n^2}(q^n+q^{-n}),\quad \alpha_{2n+1}(1;q)=0,$

$$\beta_n(1;q)=\frac{1}{(q;q)_n(q;q^2)_n};$$

(C3) $\alpha_{2n}(q;q)=(-1)^nq^{3n^2+n},\quad \alpha_{2n+1}(q;q)=(-1)^{n+1}q^{3n^2+5n+2},$

$$\beta_n(q;q)=\frac{1}{(q;q)_n(q^3;q^2)_n};$$

(C4) $\alpha_{2n}(q;q)=(-1)^nq^{3n^2+3n},\quad \alpha_{2n+1}(q;q)=(-1)^{n+1}q^{3n^2+3n},$

$$\beta_n(q;q)=\frac{q^n}{(q;q)_n(q^3;q^2)_n};$$

$$\text{(C4}^*\text{)}\quad \alpha_{2n}(q^2;q)=(-1)^nq^{3n^2+n}\frac{1-q^{4n+2}}{1-q^2},\quad \alpha_{2n+1}(q^2;q)=0,$$

$$\beta_n(q^2;q)=\frac{1}{(q;q)_n(q^3;q^2)_n};$$

$$\text{(C5)}\quad \alpha_{2n}(1;q)=(-1)^nq^{n^2}(q^n+q^{-n}),\quad \alpha_{2n+1}(1;q)=0,$$

$$\beta_n(1;q)=\frac{q^{\frac{1}{2}(n^2-n)}}{(q;q)_n(q;q^2)_n};$$

$$\text{(C6)}\quad \alpha_{2n}(q;q)=(-1)^nq^{n^2-n},\quad \alpha_{2n+1}(q;q)=(-1)^{n+1}q^{n^2+3n+2},$$

$$\beta_n(q;q)=\frac{q^{\frac{1}{2}(n^2-n)}}{(q;q)_n(q^3;q^2)_n};$$

$$\text{(C7)}\quad \alpha_{2n}(q;q)=(-1)^nq^{n^2+n},\quad \alpha_{2n+1}(q;q)=(-1)^{n+1}q^{n^2+n},$$

$$\beta_n(q;q)=\frac{q^{\frac{1}{2}(n^2+n)}}{(q;q)_n(q^3;q^2)_n};$$

$$\text{(C7}^*\text{)}\quad \alpha'_{2n}(q^2;q)=(-1)^nq^{n^2-n}\frac{1-q^{4n+2}}{1-q^2},\quad \alpha'_{2n+1}(q^2;q)=0,$$

$$\beta'_n(q^2;q)=\frac{q^{\frac{1}{2}(n^2-n)}}{(q;q)_n(q^3;q^2)_n}.$$

Note that (C4*) and (C7*) were first stated by B. Wang–L. Wang [27, p. 13, p. 14], who derived them by applying base-change method due to Lovejoy [9, Lemma 2.4] to the Bailey pairs (C4) and (C7), respectively.

We now give a proof of Theorem 2.1. We prove each of the four identities in the theorem one by one.

(*a*) **Proof of** (2.1)**.** Iterate Lemma 2.5 (with $a=1$) $r-1$ times on the Bailey pair (C1) to get a new Bailey pair $(\alpha_n^{(r-1)}(1;q),\beta_n^{(r-1)}(1;q))$ relative to 1, where

$$\alpha_{2n}^{(r-1)}(1;q)=(-1)^nq^{(4r-1)n^2}(q^n+q^{-n}),\quad \alpha_{2n+1}^{(r-1)}(1;q)=0,$$

$$\beta_n^{(r-1)}(1;q)=\sum_{n\geq N_1\geq\cdots\geq N_{r-1}\geq 0}\frac{q^{N_1^2+\cdots+N_{r-1}^2}}{(q;q)_{n-N_1}(q;q)_{N_1-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}(q;q)_{N_{r-1}}(q;q^2)_{N_{r-1}}}.$$

It follows from (2.10) that

$$\sum_{N_1\geq\cdots\geq N_{r-1}\geq 0}\frac{q^{N_1^2+\cdots+N_{r-1}^2}}{(q;q)_{N_1-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}(q;q)_{N_{r-1}}(q;q^2)_{N_{r-1}}}$$

$$= \frac{1}{(q;q)_\infty}\left(1+\sum_{n=1}^{\infty}(-1)^n q^{(4r-1)n^2}(q^n+q^{-n})\right)$$

$$= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{(4r-1)n^2+n}$$

$$\overset{(2.11)}{=} \frac{(q^{4r},q^{4r-2},q^{8r-2};q^{8r-2})_\infty}{(q;q)_\infty}.$$

(*b*) **Proof of** (2.2)**.** We consider the following two cases:

Case 1: If $j=r$, then applying Lemma 2.5 (with $a=q$) $r-1$ times to the Bailey pair (C3), we obtain a new Bailey pair:

$$\alpha_{2n}^{(r-1)}(q;q)=(-1)^n q^{(4r-1)n^2+(2r-1)n},\quad \alpha_{2n+1}^{(r-1)}(q;q)=(-1)^{n+1}q^{(4r-1)n^2+(6r-1)n+2r},$$

$$\beta_n^{(r-1)}(q;q)=\sum_{n\ge N_1\ge\cdots\ge N_{r-1}\ge 0}\frac{q^{N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{r-1}}}{(q;q)_{n-N_1}(q;q)_{N_1-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}(q;q)_{N_{r-1}}(q^3;q^2)_{N_{r-1}}}.$$

By (2.10), we derive

$$\sum_{N_1\ge\cdots\ge N_{r-1}\ge 0}\frac{q^{N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{r-1}}}{(q;q)_{N_1-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}(q;q)_{N_{r-1}}(q^3;q^2)_{N_{r-1}}}$$

$$= \frac{1}{(q^2;q)_\infty}\left(\sum_{n=0}^{\infty}(-1)^n q^{(4r-1)n^2+(2r-1)n}+\sum_{n=0}^{\infty}(-1)^{n+1}q^{(4r-1)n^2+(6r-1)n+2r}\right)$$

$$= \frac{1}{(q^2;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{(4r-1)n^2+(2r-1)n}$$

$$\overset{(2.11)}{=} \frac{(q^{6r-2},q^{2r},q^{8r-2};q^{8r-2})_\infty}{(q^2;q)_\infty}.$$

Case 2: If $1\le j<r$, then iterating Lemma 2.5 (with $a=q^2$) $r-1-j$ times on the Bailey pair (C4*) yields a new Bailey pair:

$$\alpha_{2n}^{(r-1-j)}(q^2;q)=(-1)^n q^{(4r-4j-1)n^2+(4r-4j-3)n}\frac{1-q^{4n+2}}{1-q^2},\quad \alpha_{2n+1}^{(r-j-1)}(q^2;q)=0,$$

$$\beta_n^{(r-1-j)}(q^2;q)=\sum_{n\ge N_{j+1}\ge\cdots\ge N_{r-1}\ge 0}\frac{q^{N_{j+1}^2+\cdots+N_{r-1}^2+2N_{j+1}+\cdots+2N_{r-1}}}{(q;q)_{n-N_{j+1}}(q;q)_{N_{j+1}-N_{j+2}}\cdots(q;q)_{N_{r-2}-N_{r-1}}}$$
$$\times\frac{1}{(q;q)_{N_{r-1}}(q^3;q^2)_{N_{r-1}}}. \tag{2.15}$$

Applying Lemma 2.6 (with $a=q^2$) to the Bailey pair (2.15), we get a new Bailey pair relative to q:

$$\alpha_{2n}^{(r-j)}(q;q)=(-1)^n q^{(4r-4j-1)n^2+(4r-4j-1)n},\ \alpha_{2n+1}^{(r-j)}(q;q)=(-1)^{n+1}q^{(4r-4j-1)n^2+(4r-4j-1)n},$$

$$\beta_n^{(r-j)}(q;q) = \sum_{n\geq N_{j+1}\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{N_{j+1}^2+\cdots+N_{r-1}^2+n+2N_{j+1}+\cdots+2N_{r-1}}}{(q;q)_{n-N_{j+1}}(q;q)_{N_{j+1}-N_{i+2}}\cdots(q;q)_{N_{r-2}-N_{r-1}}} \times \frac{1}{(q;q)_{N_{r-1}}(q^3;q^2)_{N_{r-1}}}. \tag{2.16}$$

Iterate Lemma 2.5 (with $a=q$) j times on (2.16) to yield

$$\alpha_{2n}^{(r)}(q;q) = (-1)^n q^{(4r-1)n^2+(4r-2j-1)n},\ \alpha_{2n+1}^{(r)}(q;q) = (-1)^{n+1} q^{(4r-1)n^2+(4r+2j-1)n+2j},$$

$$\beta_n^{(r)}(q;q) = \sum_{n\geq N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{j-1}+2N_j+\cdots+2N_{r-1}}}{(q;q)_{n-N_1}(q;q)_{N_1-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}} \times \frac{1}{(q;q)_{N_{r-1}}(q^3;q^2)_{N_{r-1}}}.$$

By means of (2.10), we get

$$\begin{aligned}
&\sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{j-1}+2N_j+\cdots+2N_{r-1}}}{(q;q)_{N_1-N_2}(q;q)_{N_2-N_3}\cdots(q;q)_{N_{r-2}-N_{r-1}}(q;q)_{N_{r-1}}(q^3;q^2)_{N_{r-1}}} \\
&= \frac{1}{(q^2;q)_\infty}\left(\sum_{n=0}^{\infty}(-1)^n q^{(4r-1)n^2+(4r-2j-1)n} + \sum_{n=0}^{\infty}(-1)^{n+1}q^{(4r-1)n^2+(4r+2j-1)n+2j}\right) \\
&= \frac{1}{(q^2;q)_\infty}\times\sum_{n=-\infty}^{\infty}(-1)^n q^{(4r-1)n^2+(4r-2j-1)n} \\
&\overset{(2.11)}{=} \frac{(q^{2j},q^{8r-2j-2},q^{8r-2};q^{8r-2})_\infty}{(q^2;q)_\infty}.
\end{aligned}$$

(*c*) **Proof of** (2.3)**.** Iterating Lemma 2.5 (with $a=1$) $r-1$ times on the Bailey pair (C5) yields a new Bailey pair $(\alpha_n^{(r-1)}(1;q),\beta_n^{(r-1)}(1;q))$ relative to 1, where

$$\alpha_{2n}^{(r-1)}(1;q) = (-1)^n q^{(4r-3)n^2}(q^n+q^{-n}), \quad \alpha_{2n+1}^{(r-1)}(1;q)=0,$$

$$\beta_n^{(r-1)}(1;q) = \sum_{n\geq N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{r-1}^2+\binom{N_{r-1}}{2}}}{(q;q)_{n-N_1}(q;q)_{N_1-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}(q;q)_{N_{r-1}}(q;q^2)_{N_{r-1}}}.$$

In view of (2.10), we get

$$\begin{aligned}
&\sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{r-1}^2+\binom{N_{r-1}}{2}}}{(q;q)_{N_1-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}(q;q)_{N_{r-1}}(q;q^2)_{N_{r-1}}} \\
&= \frac{1}{(q;q)_\infty}\left(1+\sum_{n=1}^{\infty}(-1)^n q^{(4r-3)n^2}(q^n+q^{-n})\right) \\
&= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{(4r-3)n^2+n}
\end{aligned}$$

$$\overset{(2.11)}{=} \frac{(q^{4r-4}, q^{4r-2}, q^{8r-6}; q^{8r-6})_\infty}{(q;q)_\infty}.$$

(*d*) **Proof of** (2.4)**.** We consider the following two cases:

Case 1: If $j = r$, then iterating Lemma 2.5 (with $a = q$) $r-1$ times on the Bailey pair (C6) yields

$$\alpha_{2n}^{(r-1)}(q;q) = (-1)^n q^{(4r-3)n^2+(2r-3)n}, \quad \alpha_{2n+1}^{(r-1)}(q;q) = (-1)^{n+1} q^{(4r-3)n^2+(6r-3)n+2r},$$

$$\beta_n^{(r-1)}(q;q) = \sum_{n\ge N_1\ge\cdots\ge N_{r-1}\ge 0} \frac{q^{N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{r-1}+\binom{N_{r-1}}{2}}}{(q;q)_{n-N_1}(q;q)_{N_1-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}(q;q)_{N_{r-1}}(q^3;q^2)_{N_{r-1}}}.$$

From (2.10), we obtain

$$\begin{aligned}
&\sum_{N_1\ge\cdots\ge N_{r-1}\ge 0} \frac{q^{N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{r-1}+\binom{N_{r-1}}{2}}}{(q;q)_{N_1-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}(q;q)_{N_{r-1}}(q^3;q^2)_{N_{r-1}}}\\
&= \frac{1}{(q^2;q)_\infty}\left(\sum_{n=0}^\infty (-1)^n q^{(4r-3)n^2+(2r-3)n} + \sum_{n=0}^\infty (-1)^{n+1} q^{(4r-3)n^2+(6r-3)n+2r}\right)\\
&= \frac{1}{(q^2;q)_\infty}\sum_{n=-\infty}^\infty (-1)^n q^{(4r-3)n^2+(2r-3)n}\\
&\overset{(2.11)}{=} \frac{(q^{6r-6}, q^{2r}, q^{8r-6}; q^{8r-6})_\infty}{(q^2;q)_\infty}.
\end{aligned}$$

Case 2: If $1 \le j < r$, then applying Lemma 2.5 (with $a = q^2$) $r-1-j$ times to the Bailey pair (C7*) yields the following Bailey pair:

$$\alpha_{2n}^{(r-1-j)}(q^2;q) = (-1)^n q^{(4r-4j-3)n^2+(4r-4j-5)n}\frac{1-q^{4n+2}}{1-q^2}, \quad \alpha_{2n+1}^{(r-j-1)}(q^2;q) = 0,$$

$$\begin{aligned}
\beta_n^{(r-1-j)}(q^2;q) = &\sum_{n\ge N_{j+1}\ge\cdots\ge N_{r-1}\ge 0} \frac{q^{N_{j+1}^2+\cdots+N_{r-1}^2+2N_{j+1}+\cdots+2N_{r-1}+\binom{N_{r-1}}{2}}}{(q;q)_{n-N_{j+1}}(q;q)_{N_{j+1}-N_{j+2}}\cdots(q;q)_{N_{r-2}-N_{r-1}}}\\
&\times \frac{1}{(q;q)_{N_{r-1}}(q^3;q^2)_{N_{r-1}}}. \qquad (2.17)
\end{aligned}$$

Employing Lemma 2.6 (with $a = q^2$) to the Bailey pair (2.17), we get

$$\alpha_{2n}^{(r-j)}(q;q) = (-1)^n q^{(4r-4j-3)n^2+(4r-4j-3)n}, \quad \alpha_{2n+1}^{(r-j)}(q;q) = (-1)^{n+1} q^{(4r-4j-3)n^2+(4r-4j-3)n},$$

$$\begin{aligned}
\beta_n^{(r-j)}(q;q) = &\sum_{n\ge N_{j+1}\ge\cdots\ge N_{r-1}\ge 0} \frac{q^{n+N_{j+1}^2+\cdots+N_{r-1}^2+2N_{j+1}+\cdots+2N_{r-1}+\binom{N_{r-1}}{2}}}{(q;q)_{n-N_{j+1}}(q;q)_{N_{j+1}-N_{i+2}}\cdots(q;q)_{N_{r-2}-N_{r-1}}}\\
&\times \frac{1}{(q;q)_{N_{r-1}}(q^3;q^2)_{N_{r-1}}}. \qquad (2.18)
\end{aligned}$$

Iterate Lemma 2.5 (with $a=q$) j times on (2.18) to generate the following Bailey pair:

$$\alpha_{2n}^{(r)}(q;q)=(-1)^n q^{(4r-3)n^2+(4r-2j-3)n},\quad \alpha_{2n+1}^{(r)}(q;q)=(-1)^{n+1}q^{(4r-3)n^2+(4r+2j-3)n+2j},$$

$$\beta_n^{(r)}(q;q)=\sum_{n\geq N_1\geq\cdots\geq N_{r-1}\geq 0}\frac{q^{N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{j-1}+2N_j+\cdots+2N_{r-1}+\binom{N_{r-1}}{2}}}{(q;q)_{n-N_1}(q;q)_{N_1-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}}\times\frac{1}{(q;q)_{N_{r-1}}(q^3;q^2)_{N_{r-1}}}.$$

Using (2.10), we deduce

$$\begin{aligned}&\sum_{N_1\geq\cdots\geq N_{r-1}\geq 0}\frac{q^{N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{j-1}+2N_j+\cdots+2N_{r-1}+\binom{N_{r-1}}{2}}}{(q;q)_{N_1-N_2}(q;q)_{N_2-N_3}\cdots(q;q)_{N_{r-2}-N_{r-1}}}\\&\quad\times\frac{1}{(q;q)_{N_{r-1}}(q^3;q^2)_{N_{r-1}}}\\&=\frac{1}{(q^2;q)_\infty}\left(\sum_{n=0}^{\infty}(-1)^n q^{(4r-3)n^2+(4r-2j-3)n}+\sum_{n=0}^{\infty}(-1)^{n+1}q^{(4r-3)n^2+(4r+2j-3)n+2j}\right)\\&=\frac{1}{(q^2;q)_\infty}\times\sum_{n=-\infty}^{\infty}(-1)^n q^{(4r-3)n^2+(4r-2j-3)n}\\&\overset{(2.11)}{=}\frac{(q^{2j},q^{8r-2j-6},q^{8r-6};q^{8r-6})_\infty}{(q^2;q)_\infty}.\end{aligned}$$

This completes the proof of Theorem 2.1.

2.3. **Proof of Theorem 2.2.** We first recall some Bailey pairs from Group G of Slater's list [23, p. 469], which are necessary in the proof of Theorem 2.2.

$$\text{(G1)}\quad \alpha_n(1;q)=(-1)^n q^{\frac12 n^2+\frac12\binom{n}{2}}(1+q^{n/2}),\quad \beta_n(1;q)=\frac{1}{(-q^{\frac12};q)_n(q^2;q^2)_n};$$

$$\text{(G1}^*\text{)}\quad \alpha_n(q;q)=(-1)^n q^{\frac32\binom{n+1}{2}}\frac{q^{-n}-q^{n+1}}{1-q},\quad \beta_n(q;q)=\frac{1}{(-q^{\frac12};q)_n(q^2;q^2)_n};$$

$$\text{(G2)}\quad \alpha_n(q;q)=(-1)^n q^{\frac32\binom{n+1}{2}}\frac{q^{-\frac{n}{2}}-q^{\frac{n+1}{2}}}{1-q^{\frac12}},\quad \beta_n(q;q)=\frac{1}{(-q^{\frac32};q)_n(q^2;q^2)_n};$$

$$\text{(G4)}\quad \alpha_n(1;q)=(-1)^n q^{\frac12\binom{n}{2}}(1+q^{\frac12 n}),\quad \beta_n(1;q)=\frac{(-1)^n q^{\frac12 n^2}}{(-q^{\frac12};q)_n(q^2;q^2)_n};$$

$$\text{(G4}^*\text{)}\quad \alpha_n(1;q)=(-1)^n q^{\frac12\binom{n}{2}-\frac12 n}(1+q^{\frac32 n}),\quad \beta_n(1;q)=\frac{(-1)^n q^{\frac12 n^2-n}}{(-q^{\frac12};q)_n(q^2;q^2)_n};$$

$$\text{(G4}^{**}\text{)}\quad \alpha'_n(q;q)=(-1)^n\frac{1-q^{2n+1}}{1-q}q^{\frac{1}{4}n^2-\frac{3}{4}n},\quad \beta'_n(q;q)=\frac{(-1)^nq^{\frac{1}{2}n^2-n}}{(-q^{\frac{1}{2}};q)_n(q^2;q^2)_n};$$

$$\text{(G5)}\quad \alpha_n(q;q)=(-1)^n\frac{q^{\frac{1}{2}\binom{n}{2}}(1-q^{n+\frac{1}{2}})}{1-q^{\frac{1}{2}}},\quad \beta_n(q;q)=\frac{(-1)^nq^{\frac{1}{2}n^2}}{(-q^{\frac{3}{2}};q)_n(q^2;q^2)_n}.$$

The Bailey pair (G1*) was established by B. Wang–L. Wang [26, Lemma 2.5, Eq. (2.13)] via using a generalization of a limiting case of Jackson's basic analogue of Dougall's theorem, as originally derived by Slater [23, Eq. (4.2)]. In fact, (G1*) can also be derived by applying the base-change method due to Lovejoy [10, Lemma 2.4] to the Bailey pair (G1). We note that (G4*) refers to the unlabeled Bailey pair corresponding to (G4) in [23]. Meanwhile, the Bailey pair that we term (G4**) first appeared in the work of B. Wang–L. Wang [27, Eq. (2.15)], who obtained it by applying base-change method due to Lovejoy [10, Lemma 2.4] to the Bailey pair (G4*). A typo for (G5) in [23] and [27] has been corrected here.

We are now ready to prove Theorem 2.2, establishing each of its two identities in turn.

(*a*) **Proof of** (2.5)**.** Applying Lemma 2.5 (with $a=q$) $r-1$ times to the Bailey pair (G5), we derive

$$\alpha_n^{(r-1)}(q;q)=(-1)^n\frac{q^{(r-\frac{3}{4})n^2+(r-\frac{5}{4})n}(1-q^{n+\frac{1}{2}})}{1-q^{\frac{1}{2}}},$$

$$\beta_n^{(r-1)}(q;q)=\sum_{n\ge N_1\ge\cdots\ge N_{r-1}\ge 0}\frac{(-1)^{N_{r-1}}q^{N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{r-1}+\frac{1}{2}N_{r-1}^2}}{(q;q)_{n-N_1}(q;q)_{N_1-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}(-q^{\frac{3}{2}};q)_{N_{r-1}}(q^2;q^2)_{N_{r-1}}}.$$

It follows from (2.10) that

$$\begin{aligned}&\sum_{N_1\ge\cdots\ge N_{r-1}\ge 0}\frac{(-1)^{N_{r-1}}q^{2(N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{r-1})+N_{r-1}^2}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(-q^3;q^2)_{N_{r-1}}(q^4;q^4)_{N_{r-1}}}\\&=\frac{1}{(1-q)(q^4;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^nq^{(2r-3/2)n^2+(2r-5/2)n}(1-q^{2n+1})\\&\overset{(2.11)}{=}\frac{(q,q^{4r-4},q^{4r-3};q^{4r-3})_\infty}{(1-q)(q^4;q^2)_\infty}.\end{aligned}$$

(*b*) **Proof of** (2.6)**.** There are two cases to be considered.

Case 1: If $j=r$, then applying Lemma 2.5 (with $a=1$) $r-1$ times to the Bailey pair (G4*), we obtain

$$\alpha_n^{(r-1)}(1;q)=(-1)^nq^{(r-3/4)n^2-3n/4}(1+q^{3n/2}),$$

$$\beta_n^{(r-1)}(1;q) = \sum_{n\geq N_1\geq\cdots\geq N_{r-1}} \frac{(-1)^{N_{r-1}} q^{N_1^2+\cdots+N_{r-1}^2+\frac{1}{2}N_{r-1}^2-N_{r-1}}}{(q;q)_{n-N_1}(q;q)_{N_1-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}(-q^{1/2};q)_{N_{r-1}}(q^2;q^2)_{N_{r-1}}}.$$

By (2.10), we get

$$\begin{aligned}
&\sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{(-1)^{N_{r-1}} q^{2(N_1^2+\cdots+N_{r-1}^2)+N_{r-1}^2-2N_{r-1}}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(-q;q^2)_{N_{r-1}}(q^4;q^4)_{N_{r-1}}} \\
&= \frac{1}{(q^2;q^2)_\infty}\left(1+\sum_{n=1}^{\infty}(-1)^n q^{(2r-3/2)n^2-3n/2}(1+q^{3n})\right) \\
&\overset{(2.11)}{=} \frac{(q^{2r-3},q^{2r},q^{4r-3};q^{4r-3})_\infty}{(q^2;q^2)_\infty}.
\end{aligned}$$

Case 2: If $1\leq j<r$, then applying Lemma 2.5 $r-1-j$ (with $a=q$) to the Bailey pair (G4**), we obtain

$$\begin{aligned}
\alpha_n^{(r-1-j)}(q;q) &= (-1)^n q^{(r-j-\frac{3}{4})n^2+(r-j-\frac{7}{4})n}\frac{1-q^{2n+1}}{1-q}, \\
\beta_n^{(r-1-j)}(q;q) &= \sum_{n\geq N_{j+1}\geq\cdots\geq N_{r-1}\geq 0} \frac{(-1)^{N_{r-1}} q^{N_{j+1}^2+\cdots+N_{r-1}^2+N_{j+1}+\cdots+N_{r-1}+\frac{1}{2}N_{r-1}^2-N_{r-1}}}{(q;q)_{n-N_{j+1}}(q;q)_{N_{j+1}-N_{i+2}}\cdots(q;q)_{N_{r-2}-N_{r-1}}} \\
&\quad\times\frac{1}{(-q^{1/2};q)_{N_{r-1}}(q^2;q^2)_{N_{r-1}}}. \qquad (2.19)
\end{aligned}$$

Applying Lemma 2.6 (with $a=q$) to the Bailey pair (2.19), we get

$$\begin{aligned}
\alpha_n^{(r-j)}(1;q) &= (-1)^n q^{(r-j-\frac{3}{4})n^2+(r-j-\frac{3}{4})n}(1+q^{(2j-2r+\frac{3}{2})n}), \\
\beta_n^{(r-j)}(1;q) &= \sum_{n\geq N_{j+1}\geq\cdots\geq N_{r-1}\geq 0} \frac{(-1)^{N_{r-1}} q^{N_{j+1}^2+\cdots+N_{r-1}^2+N_{j+1}+\cdots+N_{r-1}+n+\frac{1}{2}N_{r-1}^2-N_{r-1}}}{(q;q)_{n-N_{j+1}}(q;q)_{N_{j+1}-N_{i+2}}\cdots(q;q)_{N_{r-2}-N_{r-1}}} \\
&\quad\times\frac{1}{(-q^{1/2};q)_{N_{r-1}}(q^2;q^2)_{N_{r-1}}}. \qquad (2.20)
\end{aligned}$$

Applying Lemma 2.5 (with $a=1$) j times to (2.20), we obtain the Bailey pair:

$$\begin{aligned}
\alpha_n^{(r)}(1;q) &= (-1)^n q^{(r-\frac{3}{4})n^2+(r-j-\frac{3}{4})n}(1+q^{(2j-2r+\frac{3}{2})n}), \\
\beta_n^{(r)}(1;q) &= \sum_{n\geq N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{(-1)^{N_{r-1}} q^{N_1^2+\cdots+N_{r-1}^2+N_j+\cdots+N_{r-1}+\frac{1}{2}N_{r-1}^2-N_{r-1}}}{(q;q)_{n-N_1}(q;q)_{N_1-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}(-q^{1/2};q)_{N_{r-1}}(q^2;q^2)_{N_{r-1}}}.
\end{aligned}$$

Due to (2.10), we deduce that

$$\begin{aligned}
&\sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{(-1)^{N_{r-1}} q^{2(N_1^2+\cdots+N_{r-1}^2+N_j+\cdots+N_{r-1})+N_{r-1}^2-2N_{r-1}}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(-q;q^2)_{N_{r-1}}(q^4;q^4)_{N_{r-1}}} \\
&= \frac{1}{(q^2;q^2)_\infty}\left(\sum_{n=1}^{\infty}(-1)^n q^{(2r-\frac{3}{2})n^2+(2r-2j-\frac{3}{2})n}(1+q^{(4j-4r+3)n})+1\right)
\end{aligned}$$

$$\overset{(2.11)}{=} \frac{(q^{2j}, q^{4r-2j-3}, q^{4r-3}; q^{4r-3})_\infty}{(q^2; q^2)_\infty}.$$

This finishes the proof of Theorem 2.2.

2.4. **Proof of Theorem 2.3.** Here we require the following Bailey pairs, established by B. Wang–L. Wang [27, Lemma 2.5 and Lemma 2.8]:

$$\begin{aligned} &\alpha_0(1;q) = 1, \alpha_{2n}(1;q) = (-1)^n q^{3n^2}(q^{3n} + q^{-3n}), \alpha_{2n+1}(1;q) = 0, \\ &\beta_n(1;q) = \frac{q^n(1+q^{-1}) - q^{2n-1}}{(q;q)_n(q;q^2)_n}, \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} \alpha_n(q;q) &= (-1)^n q^{\frac{3}{4}(n^2-n)} \frac{1-q^{3n+\frac{3}{2}}}{1-q^{\frac{1}{2}}}, \\ \beta_n(q;q) &= \frac{q^n(1+q+q^{n+\frac{1}{2}})}{(-q^{3/2};q)_n(q;q^2)_n}. \end{aligned} \tag{2.22}$$

We now prove each of the identities (2.7) and (2.8) in Theorem 2.3 one by one.

(a) **Proof of** (2.7)**.** Iterating Lemma 2.5 (with $a=1$) $r-2$ times on the Bailey pair (2.21) yields

$$\begin{aligned} &\alpha_0^{(r-2)}(1;q) = 1,\ \alpha_{2n}^{(r-2)}(1;q) = (-1)^n q^{(4r-5)n^2}(q^{3n} + q^{-3n}),\ \alpha_{2n+1}^{(r-2)}(1;q) = 0, \\ &\beta_n^{(r-2)}(1;q) = \sum_{n \ge N_2 \ge \cdots N_{r-1} \ge 0} \frac{q^{N_2^2+N_3^2+\cdots+N_{r-1}^2+N_{r-1}}(1+q^{-1}-q^{N_{r-1}-1})}{(q;q)_{n-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}(q;q)_{N_{r-1}}(q;q^2)_{N_{r-1}}}. \end{aligned} \tag{2.23}$$

By (2.10), we have

$$\begin{aligned} &\sum_{N_1 \ge N_2 \ge \cdots N_{r-1} \ge 0} \frac{q^{N_1^2+N_2^2+N_3^2+\cdots+N_{r-1}^2+N_{r-1}}(1+q^{-1}-q^{N_{r-1}-1})}{(q;q)_{N_1-N_2}\cdots(q;q)_{N_{r-2}-N_{r-1}}(q;q)_{N_{r-1}}(q;q^2)_{N_{r-1}}} \\ &= \frac{1}{(q;q)_\infty}\left(\sum_{n=1}^{\infty}(-1)^n q^{(4r-1)n^2}(q^{3n}+q^{-3n}) + 1\right) \\ &= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{(4r-1)n^2+3n} \\ &\overset{(2.11)}{=} \frac{(q^{4r+2}, q^{4r-4}, q^{8r-2}; q^{8r-2})_\infty}{(q;q)_\infty}. \end{aligned}$$

(b) **Proof of** (2.8)**.** Iterating Lemma 2.5 (with $a=q$) $r-2$ times on the Bailey pair (2.22) gives

$$\alpha_n^{(r-2)}(q;q) = (-1)^n q^{(r-\frac{5}{4})n^2+(r-\frac{11}{4})n}\frac{1-q^{3n+\frac{3}{2}}}{1-q^{\frac{1}{2}}},$$

$$\beta_n^{(r-2)}(q;q)=\sum_{n\geq N_2\geq\cdots\geq N_{r-1}\geq 0}\frac{q^{N_2^2+N_3^2+\cdots+N_{r-1}^2+N_2+\cdots N_{r-2}+2N_{r-1}}(1+q+q^{N_{r-1}+\frac{1}{2}})}{(q;q)_{n-N_2}(q;q)_{N_2-N_3}\cdots(q;q)_{N_{r-2}-N_{r-1}}(-q^{3/2};q)_{N_{r-1}}(q^2;q^2)_{N_{r-1}}}.$$

It follows from (2.10) that

$$\sum_{N_1\geq N_2\geq\cdots\geq N_{r-1}\geq 0}\frac{q^{N_1^2+N_2^2+N_3^2+\cdots+N_{r-1}^2+N_1+N_2+\cdots N_{r-2}+2N_{r-1}}(1+q+q^{N_{r-1}+\frac{1}{2}})}{(q;q)_{n-N_2}(q;q)_{N_2-N_3}\cdots(q;q)_{N_{r-2}-N_{r-1}}(-q^{1/2};q)_{N_{r-1}+1}(q^2;q^2)_{N_{r-1}}}$$

$$=\frac{1}{(1+q^{\frac{1}{2}})(q^2;q)_\infty}\sum_{n=0}^{\infty}(-1)^nq^{(r-\frac{1}{4})n^2+(r-\frac{7}{4})n}\frac{1-q^{3n+\frac{3}{2}}}{1-q^{\frac{1}{2}}}$$

$$=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^nq^{(r-\frac{1}{4})n^2+(r-\frac{7}{4})n}$$

$$\overset{(2.11)}{=}\frac{(q^{\frac{3}{2}},q^{2r-2},q^{2r-\frac{1}{2}};q^{2r-\frac{1}{2}})_\infty}{(q;q)_\infty}.$$

This completes the proof of Theorem 2.3.

3. Proofs of Theorems 1.5, 1.10, 1.12

Armed with identities in Theorems 2.1, 2.2 and 2.3, we are ready to prove Theorems 1.5, 1.10 and 1.12. Throughout the entire article, we let $N_i=n_{i+1}+\cdots+n_r$ for $1\leq i\leq r-1$ and $N_i=0$ for $i>r-1$.

3.1. **Proof of Theorem 1.5.** We prove each of the six identities in the theorem one by one.

(1) **Proof of** (1.7)**.** It relies on (2.1) from Theorem 2.1 and the following multi-sum Rogers–Ramanujan identity from [26, Eq. (1.16)]:

$$\sum_{N_1\geq\cdots\geq N_{r-1}\geq 0}\frac{q^{2(N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{r-1})}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^4;q^4)_{N_{r-1}}(-q^3;q^2)_{N_{r-1}}}$$

$$=\frac{(q,q^{4r-2},q^{4r-1};q^{4r-1})_\infty}{(1-q)(q^4;q^2)_\infty}. \tag{3.1}$$

Let AD be the symmetric matrix given by (1.4). Then we have

$$\frac{1}{2}\boldsymbol{n}^TAD\boldsymbol{n}$$

$$=n_1^2+\sum_{j=2}^{r-1}\left(2(j-1)n_j^2\right)+\frac{r-1}{2}n_r^2+2\sum_{2\leq j<k\leq r-1}(2(j-1)n_jn_k)+n_1n_r$$

$$+2\sum_{j=2}^{r-1}((j-1)n_jn_r)$$

$$=n_1{}^2+n_1n_r+2\left(\left(n_2+\cdots+n_{r-1}+\frac{n_r}{2}\right)^2+\cdots+\left(n_{r-1}+\frac{n_r}{2}\right)^2+\frac{n_r^2}{4}\right). \quad (3.2)$$

Since $\boldsymbol{b}_0=(0,0,\cdots,0,0)^T$, the multiple summation of (1.7) simplifies to:

$$F:=\sum_{n=(n_1,\dots,n_r)^T\in\mathbb{N}^r}\frac{q^{\frac{1}{2}\boldsymbol{n}^TAD\boldsymbol{n}+\boldsymbol{n}^T\boldsymbol{b}_0}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{n_r}}$$

$$=\sum_{n_1,n_2,\dots,n_r\geq 0}\frac{q^{n_1{}^2+n_1n_r+2\left(\left(n_2+\cdots+\frac{n_r}{2}\right)^2+\cdots+\left(n_{r-1}+\frac{n_r}{2}\right)^2+\frac{n_r^2}{4}\right)}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{n_r}}.$$

For $\sigma\in\{0,1\}$, let

$$F_\sigma=\sum_{\substack{n_r\equiv\sigma \pmod 2\\ n_1,n_2,\dots,n_r\geq 0}}\frac{q^{n_1{}^2+n_1n_r+2\left(\left(n_2+\cdots+n_{r-1}+\frac{n_r}{2}\right)^2+\cdots+\left(n_{r-1}+\frac{n_r}{2}\right)^2+\frac{n_r^2}{4}\right)}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{n_r}}$$

$$=\sum_{n_1,\dots,n_r\geq 0}\frac{q^{n_1{}^2+n_1(2n_r+\sigma)+2\left(\left(n_2+\cdots+n_{r-1}+\frac{2n_r+\sigma}{2}\right)^2+\cdots+\left(n_{r-1}+\frac{2n_r+\sigma}{2}\right)^2+\frac{(2n_r+\sigma)^2}{4}\right)}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{2n_r+\sigma}}. \quad (3.3)$$

It is straightforward that

$$F=F_0+F_1. \quad (3.4)$$

We proceed to simplify F_0 and F_1 respectively. To this end, we invoke Euler's q-exponential identity [1, Corollary 2.2]:

$$\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}z^n}{(q;q)_n}=(-z;q)_\infty. \quad (3.5)$$

First, for F_0, we find that

$$F_0=\sum_{n_1,\dots,n_r\geq 0}\frac{q^{n_1{}^2+2n_1n_r+2((n_2+\cdots+n_{r-1}+n_r)^2+\cdots+(n_{r-1}+n_r)^2+n_r^2)}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{2n_r}}$$

$$=\sum_{N_1\geq\cdots\geq N_{r-1}\geq 0}\frac{q^{2(N_1^2+N_2^2+\cdots+N_{r-1}^2)}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q;q)_{2N_{r-1}}}\sum_{n_1=0}^{\infty}\frac{q^{n_1{}^2+2n_1N_{r-1}}}{(q^2;q^2)_{n_1}}$$

$$\overset{(3.5)}{=}\sum_{N_1\geq\cdots\geq N_{r-1}\geq 0}\frac{q^{2(N_1^2+\cdots+N_{r-1}^2)}(-q^{2N_{r-1}+1};q^2)_\infty}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q;q)_{2N_{r-1}}}$$

$$=\sum_{N_1\geq\cdots\geq N_{r-1}\geq 0}\frac{(-q;q^2)_\infty q^{2(N_1^2+\cdots+N_{r-1}^2)}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^2;q^2)_{N_{r-1}}(q^2;q^4)_{N_{r-1}}}. \quad (3.6)$$

Similarly, for F_1, we obtain

$$F_1=\sum_{n_1,\dots,n_r\geq 0}\frac{q^{n_1{}^2+n_1(2n_r+1)+2\left(\left(n_2+\cdots+n_{r-1}+\frac{2n_r+1}{2}\right)^2+\cdots+\left(n_{r-1}+\frac{2n_r+1}{2}\right)^2+\frac{(2n_r+1)^2}{4}\right)}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{2n_r+1}}$$

$$
\begin{aligned}
&= \sum_{n_1,N_1,\ldots,N_{r-1}\geq 0} \frac{q^{{n_1}^2+n_1(2N_{r-1}+1)+2\left(\left(N_1+\frac{1}{2}\right)^2+\cdots+\left(N_{r-1}+\frac{1}{2}\right)^2\right)}}{(q^2;q^2)_{n_1}(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q;q)_{2N_{r-1}+1}}\\
&= q^{\frac{r-1}{2}} \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{r-1})}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q;q)_{2N_{r-1}+1}} \sum_{n_1=0}^{\infty} \frac{q^{n_1^2+n_1(2N_{r-1}+1)}}{(q^2;q^2)_{n_1}}\\
&\overset{(3.5)}{=} q^{\frac{r-1}{2}} \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{r-1})}(-q^{2N_{r-1}+2};q^2)_\infty}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q;q)_{2N_{r-1}+1}}\\
&= \frac{q^{\frac{r-1}{2}}(-q^2;q^2)_\infty}{(1-q)} \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{r-1})}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^4;q^4)_{N_{r-1}}(q^3;q^2)_{N_{r-1}}}.
\end{aligned}
\tag{3.7}
$$

Next, substituting (2.1) with q replaced by q^2 into (3.6), we have

$$
\begin{aligned}
F_0 &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(q^{8r-4},q^{8r},q^{16r-4};q^{16r-4})_\infty\\
&= \frac{(q^{8r-4},q^{8r},q^{16r-4};q^{16r-4})_\infty}{(q,q^3,q^4;q^4)_\infty}.
\end{aligned}
\tag{3.8}
$$

Likewise, substituting (3.1) with q replaced by $-q$ into (3.7), we find

$$
\begin{aligned}
F_1 &= q^{\frac{r-1}{2}}\frac{(-q^2;q^2)_\infty}{1-q}\frac{(-q,q^{4r-2},-q^{4r-1};-q^{4r-1})_\infty}{(1+q)(q^4;q^2)_\infty}\\
&= q^{\frac{r-1}{2}}\frac{(-q,q^{4r-2},-q^{4r-1};-q^{4r-1})_\infty}{(q^2,q^2,q^4;q^4)_\infty}.
\end{aligned}
\tag{3.9}
$$

Finally, adding (3.8) and (3.9), by (3.4), we obtain (1.7).

(2) **Proof of** (1.8)**.** It is based on (2.2) from Theorem 2.1 and the following multi-sum Rogers–Ramanujan identity from [26, Eq. (1.14)]:

$$
\begin{aligned}
&\sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+\cdots+N_{r-1}^2+N_j+\cdots+N_{r-1})}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^4;q^4)_{N_{r-1}}(-q;q^2)_{N_{r-1}}}\\
&= \frac{(q^{2j},q^{4r-2j-1},q^{4r-1};q^{4r-1})_\infty}{(q^2;q^2)_\infty}.
\end{aligned}
\tag{3.10}
$$

Let AD be the symmetric matrix defined by (1.4), and let $\boldsymbol{b}_j=(1,0,\ldots,0,2,4,\cdots,2(r-1-j),r-j)^T$ for $1\leq j\leq r$. By virtue of (3.2), we deduce that

$$
\begin{aligned}
&\frac{1}{2}\boldsymbol{n}^{\mathrm{T}}AD\boldsymbol{n}+\boldsymbol{n}^{\mathrm{T}}\boldsymbol{b}_j\\
=&n_1^2+n_1n_r+2\left(\left(n_2+\cdots+n_{r-1}+\frac{n_r}{2}\right)^2+\cdots+\left(n_{r-1}+\frac{n_r}{2}\right)^2+\frac{n_r^2}{4}\right)\\
&+n_1+2\left(\left(n_{j+1}+\cdots+n_{r-1}+\frac{n_r}{2}\right)+\cdots+\left(n_{r-1}+\frac{n_r}{2}\right)+\frac{n_r}{2}\right).
\end{aligned}
$$

Hence the multiple summation of (1.8) can be simplified to:

$$H = \sum_{\boldsymbol{n}=(n_1,\dots,n_r)^T\in\mathbb{N}^r} \frac{q^{\frac{1}{2}\boldsymbol{n}^{\mathrm{T}}AD\boldsymbol{n}+\boldsymbol{n}^{\mathrm{T}}\boldsymbol{b}_j}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{n_r}}$$

$$= \sum_{n_2,\dots,n_r\geq 0} \frac{q^{2\left(\left(n_2+\cdots+n_{r-1}+\frac{n_r}{2}\right)^2+\cdots+\left(n_{r-1}+\frac{n_r}{2}\right)^2+\frac{1}{4}n_r^2\right)}}{(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{n_r}}$$

$$\times \sum_{n_1=0}^{\infty} \frac{q^{n_1^2+n_1n_r+n_1+2\left(n_{j+1}+\cdots+n_{r-1}+\frac{n_r}{2}+\cdots+n_{r-1}+\frac{n_r}{2}+\frac{n_r}{2}\right)}}{(q^2;q^2)_{n_1}}.$$

For $\sigma\in\{0,1\}$, we define H_σ corresponding to $n_r\equiv\sigma \pmod 2$:

$$H_\sigma = \sum_{n_2,\dots,n_r\geq 0} \frac{q^{2\left(\left(n_2+\cdots+n_{r-1}+\frac{2n_r+\sigma}{2}\right)^2+\cdots+\left(n_{r-1}+\frac{2n_r+\sigma}{2}\right)^2+\frac{(2n_r+\sigma)^2}{4}\right)}}{(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{2n_r+\sigma}}$$

$$\times \sum_{n_1=0}^{\infty} \frac{q^{n_1^2+n_1(2n_r+\sigma)+n_1+2\left((n_{j+1}+\cdots+n_{r-1}+\frac{2n_r+\sigma}{2})+\cdots+(n_{r-1}+\frac{2n_r+\sigma}{2})+\frac{2n_r+\sigma}{2}\right)}}{(q^2;q^2)_{n_1}}.$$

It is immediate that

$$H = H_0 + H_1. \tag{3.11}$$

Next, we simplify H_0 as follows:

$$H_0 = \sum_{n_2,\dots,n_r\geq 0} \frac{q^{2((n_2+\cdots+n_r)^2+\cdots+(n_{r-1}+n_r)^2+n_r^2)}}{(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{2n_r}}$$

$$\times \sum_{n_1=0}^{\infty} \frac{q^{n_1^2+2n_1n_r+n_1+2((n_{j+1}+\cdots+n_r)+\cdots+(n_{r-1}+n_r)+n_r)}}{(q^2;q^2)_{n_1}}$$

$$= \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+\cdots+N_{r-1}^2+N_j+\cdots+N_{r-1})}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q;q)_{2N_{r-1}}} \sum_{n_1=0}^{\infty} \frac{q^{n_1^2+2n_1N_{r-1}+n_1}}{(q^2;q^2)_{n_1}}$$

$$\overset{(3.5)}{=} \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+\cdots+N_{r-1}^2+N_j+\cdots+N_{r-1})}(-q^{2N_{r-1}+2};q^2)_\infty}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q;q)_{2N_{r-1}}}$$

$$= \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+\cdots+N_{r-1}^2+N_j+\cdots+N_{r-1})}(-q^2;q^2)_\infty}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^4;q^4)_{N_{r-1}}(q;q^2)_{N_{r-1}}}. \tag{3.12}$$

Similarly, the expression for H_1 is derived as

$$H_1 = \sum_{n_2,\dots,n_r\geq 0} \frac{q^{2\left(\left(n_2+\cdots+n_{r-1}+\frac{2n_r+1}{2}\right)^2+\cdots+\left(n_{r-1}+\frac{2n_r+1}{2}\right)^2+\frac{(2n_r+1)^2}{4}\right)}}{(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{2n_r+1}}$$

$$\times \sum_{n_1=0}^{\infty} \frac{q^{n_1^2+n_1(2n_r+1)+n_1+2\left(\left(n_{j+1}+\cdots+n_{r-1}+\frac{2n_r+1}{2}\right)+\cdots+\left(n_{r-1}+\frac{2n_r+1}{2}\right)+\frac{2n_r+1}{2}\right)}}{(q^2;q^2)_{n_1}}$$

$$= \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2\left(\left(N_1+\frac{1}{2}\right)^2+\cdots+\left(N_{r-1}+\frac{1}{2}\right)^2+N_j+\cdots+N_{r-1}+\frac{r-j}{2}\right)}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q;q)_{2N_{r-1}+1}} \sum_{n_1=0}^{\infty} \frac{q^{n_1^2+n_1(2N_{r-1}+1)+n_1}}{(q^2;q^2)_{n_1}}$$

$$\overset{(3.5)}{=} q^{\frac{3r-2j-1}{2}} \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2\left(N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{j-1}+2N_j+\cdots+2N_{r-1}\right)}(-q^{2N_{r-1}+3};q^2)_\infty}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q;q)_{2N_{r-1}+1}}$$

$$= \frac{q^{\frac{3r-2j-1}{2}}(-q^3;q^2)_\infty}{(1-q)} \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{j-1}+2N_j+\cdots+2N_{r-1})}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^2;q^2)_{N_{r-1}}(q^6;q^4)_{N_{r-1}}}. \tag{3.13}$$

Substituting the identity (3.10) (for $1\leq j\leq r$) with $q\mapsto -q$ into (3.12), we obtain

$$\begin{aligned} H_0 &= \frac{(-q^2;q^2)_\infty(q^{2j},-q^{4r-2j-1},-q^{4r-1};-q^{4r-1})_\infty}{(q^2;q^2)_\infty} \\ &= \frac{(q^{2j},-q^{4r-2j-1},-q^{4r-1};-q^{4r-1})_\infty}{(q^2,q^2,q^4;q^4)_\infty}. \end{aligned} \tag{3.14}$$

Likewise, substituting (2.2) (for $1\leq j\leq r$) with $q\mapsto q^2$ into (3.13), we get

$$\begin{aligned} H_1 &= q^{\frac{3r-2j-1}{2}} \frac{(-q^3;q^2)_\infty(q^{4j},q^{16r-4j-4},q^{16r-4};q^{16r-4})_\infty}{(1-q)(q^4;q^2)_\infty} \\ &= q^{\frac{3r-2j-1}{2}} \frac{(q^{4j},q^{16r-4j-4},q^{16r-4};q^{16r-4})_\infty}{(q,q^3,q^4;q^4)_\infty}. \end{aligned} \tag{3.15}$$

Combining (3.11), (3.14) and (3.15), we arrive at the identity (1.8).

(3) **Proof of** (1.9)**.** It depends on (2.3) from Theorem 2.1 and (2.5) from Theorem 2.2.

Let BD be the symmetric matrix given by (1.5), and $\boldsymbol{b}_0 = (0,0,\ldots,0,-\frac{1}{2})^T$. Similar to (3.2), we derive that

$$\frac{1}{2}\boldsymbol{n}^{\mathrm{T}}BD\boldsymbol{n}+\boldsymbol{n}^{\mathrm{T}}\boldsymbol{b}_0$$

$$= n_1^2+n_1n_r+2\left(\left(n_2+\cdots+n_{r-1}+\frac{n_r}{2}\right)^2+\cdots+\left(n_{r-1}+\frac{n_r}{2}\right)^2+\frac{n_r^2}{4}\right)+\frac{n_r^2}{4}-\frac{n_r}{2}.$$

Hence the multiple summation of (1.9) can be written as:

$$\begin{aligned} I &= \sum_{\boldsymbol{n}=(n_1,\ldots,n_r)^T\in\mathbb{N}^r} \frac{q^{\frac{1}{2}\boldsymbol{n}^{\mathrm{T}}BD\boldsymbol{n}+\boldsymbol{n}^{\mathrm{T}}\boldsymbol{b}_0}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{n_r}} \\ &= \sum_{n_1,\ldots,n_r\geq 0} \frac{q^{n_1^2+n_1n_r}q^{2\left(\left(n_2+\cdots+n_{r-1}+\frac{1}{2}n_r\right)^2+\cdots+\left(n_{r-1}+\frac{1}{2}n_r\right)^2+\frac{1}{4}n_r^2\right)+\frac{1}{4}n_r^2-\frac{1}{2}n_r}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{n_r}}. \end{aligned}$$

For $\sigma \in \{0,1\}$, we set

$$I_\sigma = \sum_{n_1,\dots,n_r\geq 0} \frac{q^{2\left(\left(n_2+\cdots+n_{r-1}+\frac{1}{2}(2n_r+\sigma)\right)^2+\cdots+\left(n_{r-1}+\frac{1}{2}(2n_r+\sigma)\right)^2+\frac{1}{4}(2n_r+\sigma)^2\right)}}{(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{2n_r+\sigma}} \times \frac{q^{n_1^2+n_1(2n_r+\sigma)}q^{\frac{1}{4}(2n_r+\sigma)^2-\frac{1}{2}(2n_r+\sigma)}}{(q^2;q^2)_{n_1}}.$$

It follows that

$$I = I_0 + I_1. \tag{3.16}$$

We proceed to simplify I_0:

$$\begin{aligned} I_0 &= \sum_{n_1,\dots,n_r\geq 0} \frac{q^{n_1^2+2n_1n_r}q^{2((n_2+\cdots+n_r)^2+\cdots+(n_{r-1}+n_r)^2+n_r^2)+n_r^2-n_r}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{2n_r}} \\ &= \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+N_2^2+\cdots+N_{r-1}^2)+N_{r-1}^2-N_{r-1}}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q;q)_{2N_{r-1}}} \sum_{n_1=0}^{\infty} \frac{q^{n_1^2+2n_1N_{r-1}}}{(q^2;q^2)_{n_1}} \\ &\overset{(3.5)}{=} \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+\cdots+N_{r-1}^2)+N_{r-1}^2-N_{r-1}}(-q^{2N_{r-1}+1};q^2)_\infty}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q;q)_{2N_{r-1}}} \\ &= (-q;q^2)_\infty \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+\cdots+N_{r-1}^2)}q^{N_{r-1}^2-N_{r-1}}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^2;q^2)_{N_{r-1}}(q^2;q^4)_{N_{r-1}}}. \end{aligned} \tag{3.17}$$

Similarly, the simplification of I_1 yields

$$\begin{aligned} I_1 &= \sum_{n_1,\dots,n_r\geq 0} \frac{q^{n_1^2+n_1(2n_r+1)+2\left(\left(n_2+\cdots+n_{r-1}+\frac{2n_r+1}{2}\right)^2+\cdots+\frac{(2n_r+1)^2}{4}\right)+\frac{(2n_r+1)^2}{4}-\frac{2n_r+1}{2}}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{2n_r+1}} \\ &= \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2\left(\left(N_1+\frac{1}{2}\right)^2+\cdots+\left(N_{r-1}+\frac{1}{2}\right)^2\right)+\frac{(2N_{r-1}+1)^2}{4}-\frac{2N_{r-1}+1}{2}}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q;q)_{2N_{r-1}+1}} \sum_{n_1=0}^{\infty} \frac{q^{n_1^2+n_1(2N_{r-1}+1)}}{(q^2;q^2)_{n_1}} \\ &\overset{(3.5)}{=} q^{\frac{2r-3}{4}} \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{r-1})+N_{r-1}^2}(-q^{2N_{r-1}+2};q^2)_\infty}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q;q)_{2N_{r-1}+1}} \\ &= q^{\frac{2r-3}{4}} \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{r-1})}q^{N_{r-1}^2}(-q^2;q^2)_\infty}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^3;q^2)_{N_{r-1}}(q^4;q^4)_{N_{r-1}}(1-q)}. \end{aligned} \tag{3.18}$$

Substituting (2.3) with $q \mapsto q^2$ into (3.17), we get

$$I_0 = \frac{(-q;q^2)_\infty(q^{8r-8},q^{8r-4},q^{16r-12};q^{16r-12})_\infty}{(q^2;q^2)_\infty}$$

$$= \frac{(q^{8r-8}, q^{8r-4}, q^{16r-12}; q^{16r-12})_\infty}{(q, q^3, q^4; q^4)_\infty}. \tag{3.19}$$

Substituting (2.5) with $q \mapsto -q$ into (3.18), we obtain

$$\begin{aligned} I_1 &= q^{\frac{2r-3}{4}} \frac{(-q^2;q^2)_\infty(-q, q^{4r-4}, -q^{4r-3}; -q^{4r-3})_\infty}{(1-q)(1+q)(q^4;q^2)_\infty} \\ &= q^{\frac{2r-3}{4}} \frac{(-q, q^{4r-4}, -q^{4r-3}; -q^{4r-3})_\infty}{(q^2, q^2, q^4; q^4)_\infty}. \end{aligned} \tag{3.20}$$

By combining (3.16), (3.19) and (3.20), we establish the identity (1.9).

(4) **Proof of** (1.10)**.** The argument utilizes (2.4) from Theorem 2.1 and (2.6) from Theorem 2.2.

Let BD be the symmetric matrix given by (1.5), and let $\boldsymbol{b}_j = (1, 0, \ldots, 0, 2, 4, \ldots, 2(r-j-1), r-j-1)^T$. We deduce that

$$\begin{aligned} &\frac{1}{2}\boldsymbol{n}^{\mathrm{T}} BD\boldsymbol{n} + \boldsymbol{n}^{\mathrm{T}}\boldsymbol{b}_j \\ =&n_1^2 + n_1 n_r + 2\left(\left(n_2 + \cdots + n_{r-1} + \frac{n_r}{2}\right)^2 + \cdots + \left(n_{r-1} + \frac{n_r}{2}\right)^2 + \frac{n_r^2}{4}\right) + \frac{n_r^2}{4} \\ &+ n_1 - n_r + 2\left(\left(n_{j+1} + \cdots + n_{r-1} + \frac{n_r}{2}\right) + \cdots + \left(n_{r-1} + \frac{n_r}{2}\right) + \frac{n_r}{2}\right). \end{aligned}$$

Thus, the multiple sum in (1.10) takes the form

$$\begin{aligned} K &= \sum_{\boldsymbol{n}=(n_1,\ldots,n_r)^T \in \mathbb{N}^r} \frac{q^{\frac{1}{2}\boldsymbol{n}^{\mathrm{T}} BD\boldsymbol{n} + \boldsymbol{n}^{\mathrm{T}}\boldsymbol{b}_j}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{n_r}} \\ &= \sum_{n_1,\ldots,n_r \geq 0} \frac{q^{2\left(\left(n_2+\cdots+n_{r-1}+\frac{1}{2}n_r\right)^2+\cdots+\left(n_{r-1}+\frac{1}{2}n_r\right)^2+\frac{1}{4}n_r^2\right)+\frac{1}{4}n_r^2-n_r}}{(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{n_r}} \\ &\quad \times \frac{q^{n_1^2+n_1n_r+n_1+2\left(\left(n_{j+1}+\cdots+n_{r-1}+\frac{1}{2}n_r\right)+\cdots+\left(n_{r-1}+\frac{1}{2}n_r\right)+\frac{1}{2}n_r\right)}}{(q^2;q^2)_{n_1}}. \end{aligned}$$

For $\sigma \in \{0, 1\}$, we set

$$\begin{aligned} K_\sigma = \sum_{n_1,\ldots,n_r \geq 0} &\frac{q^{2\left(\left(n_2+\cdots+n_{r-1}+\frac{1}{2}(2n_r+\sigma)\right)^2+\cdots+\left(n_{r-1}+\frac{1}{2}(2n_r+\sigma)\right)^2+\frac{1}{4}(2n_r+\sigma)^2\right)+\frac{1}{4}(2n_r+\sigma)^2-(2n_r+\sigma)}}{(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{2n_r+\sigma}} \\ &\times \frac{q^{n_1^2+n_1(2n_r+\sigma)+n_1+2\left(\left(n_{j+1}+\cdots+n_{r-1}+\frac{1}{2}(2n_r+\sigma)\right)+\cdots+\left(n_{r-1}+\frac{1}{2}(2n_r+\sigma)\right)+\frac{1}{2}(2n_r+\sigma)\right)}}{(q^2;q^2)_{n_1}}. \end{aligned}$$

It is clear that

$$K = K_0 + K_1. \tag{3.21}$$

Next, we simplify K_0 as follows:

$$K_0 = \sum_{n_1,\dots,n_r\geq 0} \frac{q^{2((n_2+\cdots+n_r)^2+\cdots+(n_{r-1}+n_r)^2+n_r^2)+n_r^2-2n_r}}{(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{2n_r}}$$
$$\times \frac{q^{n_1^2+2n_1n_r+n_1+2((n_{j+1}+\cdots+n_r)+\cdots+(n_{r-1}+n_r)+n_r)}}{(q^2;q^2)_{n_1}}$$
$$= \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+\cdots+N_{r-1}^2+N_j+\cdots+N_{r-1})+N_{r-1}^2-2N_{r-1}}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q;q)_{2N_{r-1}}} \sum_{n_1=0}^{\infty} \frac{q^{n_1^2+2n_1N_{r-1}+n_1}}{(q^2;q^2)_{n_1}}$$
$$\overset{(3.5)}{=} \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+\cdots+N_{r-1}^2+N_j+\cdots+N_{r-1})+N_{r-1}^2-2N_{r-1}}(-q^{2N_{r-1}+2};q^2)_\infty}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q;q)_{2N_{r-1}}}$$
$$= \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+\cdots+N_{r-1}^2+N_j+\cdots+N_{r-1})+N_{r-1}^2-2N_{r-1}}(-q^2;q^2)_\infty}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^4;q^4)_{N_{r-1}}(q;q^2)_{N_{r-1}}}. \tag{3.22}$$

The simplification of K_1 gives

$$K_1 = \sum_{n_1,\dots,n_r\geq 0} \frac{q^{2\left((n_2+\cdots+n_{r-1}+\frac{2n_r+1}{2})^2+\cdots+(n_{r-1}+\frac{2n_r+1}{2})^2+\frac{(2n_r+1)^2}{4}\right)+\frac{(2n_r+1)^2}{4}-(2n_r+1)}}{(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{2n_r+1}}$$
$$\times \frac{q^{n_1^2+n_1(2n_r+1)+n_1+2\left(\left(n_{j+1}+\cdots+n_{r-1}+\frac{2n_r+1}{2}\right)+\cdots+\left(n_{r-1}+\frac{2n_r+1}{2}\right)+\frac{2n_r+1}{2}\right)}}{(q^2;q^2)_{n_1}}$$
$$= \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2\left(\left(N_1+\frac12\right)^2+\cdots+\left(N_{r-1}+\frac12\right)^2+N_j+\cdots+N_{r-1}+\frac{r-j}{2}\right)+N_{r-1}^2-N_{r-1}-\frac34}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q;q)_{2N_{r-1}+1}}$$
$$\sum_{n_1=0}^{\infty} \frac{q^{n_1^2+n_1(2N_{r-1}+1)+n_1}}{(q^2;q^2)_{n_1}}$$
$$\overset{(3.5)}{=} (-q^{2N_{r-1}+3};q^2)_\infty$$
$$\times \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2\left(N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{j-1}+2N_j+\cdots+2N_{r-1}+\frac{3r-2j-1}{4}\right)+N_{r-1}^2-N_{r-1}-\frac34}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q;q)_{2N_{r-1}+1}}$$
$$= \frac{q^{\frac{6r-4j-5}{4}}(-q^3;q^2)_\infty}{(1-q)}$$
$$\times \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{j-1}+2N_j+\cdots+2N_{r-1})+N_{r-1}^2-N_{r-1}}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^2;q^2)_{N_{r-1}}(q^6;q^4)_{N_{r-1}}}. \tag{3.23}$$

Substituting (2.6) (for $1\leq j\leq r$) with $q\mapsto -q$ into (3.22), we obtain

$$K_0 = \frac{(-q^2;q^2)_\infty(q^{2j},-q^{4r-2j-3},-q^{4r-3};-q^{4r-3})_\infty}{(q^2;q^2)_\infty}$$

$$= \frac{(q^{2j}, -q^{4r-2j-3}, -q^{4r-3}; -q^{4r-3})_\infty}{(q^2, q^2, q^4; q^4)_\infty}. \tag{3.24}$$

Substituting (2.4) (for $1 \leq j \leq r$) with $q \mapsto q^2$ into (3.22), we derive that

$$\begin{aligned} K_1 &= q^{\frac{6r-4j-5}{4}} \frac{(-q^3; q^2)_\infty (q^{4j}, q^{16r-4j-12}, q^{16r-12}; q^{16r-12})_\infty}{(1-q)(q^4; q^2)_\infty} \\ &= q^{\frac{6r-4j-5}{4}} \frac{(q^{4j}, q^{16r-4j-12}, q^{16r-12}; q^{16r-12})_\infty}{(q, q^3, q^4; q^4)_\infty}. \end{aligned} \tag{3.25}$$

Finally, substituting (3.24) and (3.25) into (3.21), we establish the identity (1.10).

(5) **Proof of** (1.11)**.** This identity is equivalent to that established by B. Wang–L. Wang in [26, Corollary 4.3]. We provide an explanation for completeness.

Let $A^\vee D^\vee$ be the symmetric matrix defined by (1.6), and let $\boldsymbol{b}_0 = (1, 1, 2, 3, \ldots, r-1)^T_{1\times r}$. We have

$$\begin{aligned} &\frac{1}{2}\boldsymbol{n}^T A^\vee D^\vee \boldsymbol{n} + \boldsymbol{n}^T \boldsymbol{b}_0 \\ =&\frac{1}{2}n_1^2 + n_1 n_r + \Big((n_2 + \cdots + n_r)^2 + \cdots + (n_{r-1} + n_r)^2 + n_r^2\Big) \\ &+ n_1 + (n_2 + \cdots + n_r) + \cdots + (n_{r-1} + n_r) + n_r. \end{aligned}$$

It follows that the multiple sum in (1.11) becomes

$$\begin{aligned} &\sum_{\boldsymbol{n}=(n_1,\ldots,n_r)^T \in \mathbb{N}^r} \frac{q^{\frac{1}{2}\boldsymbol{n}^T A^\vee D^\vee \boldsymbol{n} + \boldsymbol{n}^T \boldsymbol{b}_0}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{r-1}}(q^2;q^2)_{n_r}} \\ =&\sum_{n_1,\ldots,n_r \geq 0} \frac{q^{\frac{1}{2}n_1^2 + n_1 n_r + \left((n_2+\cdots+n_r)^2 + \cdots + (n_{r-1}+n_r)^2 + n_r^2\right) + n_1 + (n_2+\cdots+n_r) + \cdots + (n_{r-1}+n_r) + n_r}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{r-1}}(q^2;q^2)_{n_r}} \\ =&\sum_{n_1,\ldots,n_r \geq 0} \frac{q^{\frac{1}{2}n_1^2 + n_1 N_{r-1} + \left(N_1^2 + \cdots + N_{r-1}^2\right) + n_1 + N_1 + \cdots + N_{r-1}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{r-1}}(q^2;q^2)_{n_r}} \\ =&\frac{(-q^{\frac{1}{2}}; q)_\infty (q^{\frac{1}{2}}, q^{2r-1}, q^{2r-\frac{1}{2}}; q^{2r-\frac{1}{2}})_\infty}{(q;q)_\infty} \\ =&\frac{(q^{\frac{1}{2}}, q^{2r-1}, q^{2r-\frac{1}{2}}; q^{2r-\frac{1}{2}})_\infty}{(q^{\frac{1}{2}}, q^{\frac{3}{2}}, q^2; q^2)_\infty}, \end{aligned}$$

where the second last identity is due to B. Wang–L. Wang [26, Corollary 4.3].

(6) **Proof of** (1.12)**.** This identity is equivalent to the one established by B. Wang and L. Wang in [26, Corollary 4.2]. For the sake of completeness, we provide an explanation herein.

Let $A^\vee D^\vee$ be the symmetric matrix defined by (1.6), and let $\boldsymbol{b}_j = (0, \ldots, 0, 1, 2, \ldots, r-j)^T_{1\times r}$. We have

$$\begin{aligned}
&\frac{1}{2}\boldsymbol{n}^T A^\vee D^\vee \boldsymbol{n} + \boldsymbol{n}^T \boldsymbol{b}_j \\
=&\frac{1}{2}n_1^2 + n_1 n_r + \Big((n_2 + \cdots + n_r)^2 + \cdots + (n_{r-1} + n_r)^2 + n_r^2\Big) \\
&+ (n_{j+1} + \cdots + n_r) + \cdots + (n_{r-1} + n_r) + n_r.
\end{aligned}$$

We consider the multiple summation of (1.12),

$$\begin{aligned}
&\sum_{\boldsymbol{n}=(n_1,\ldots,n_r)^T\in\mathbb{N}^r} \frac{q^{\frac{1}{2}\boldsymbol{n}^T A^\vee D^\vee \boldsymbol{n} + \boldsymbol{n}^T \boldsymbol{b}_j}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{r-1}}(q^2;q^2)_{n_r}} \\
=&\sum_{n_1,\ldots,n_r\geq 0} \frac{q^{\frac{1}{2}n_1^2+n_1n_r+\left((n_2+\cdots+n_r)^2+\cdots+(n_{r-1}+n_r)^2+n_r^2\right)+(n_{j+1}+\cdots+n_r)+\cdots+(n_{r-1}+n_r)+n_r}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{r-1}}(q^2;q^2)_{n_r}} \\
=&\sum_{n_1,\ldots,n_r\geq 0} \frac{q^{\frac{1}{2}n_1^2+n_1N_{r-1}+\left(N_1^2+\cdots+N_{r-1}^2\right)+N_j+\cdots+N_{r-1}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{r-1}}(q^2;q^2)_{n_r}} \\
=&\frac{(-q^{\frac{1}{2}};q)_\infty (q^j, q^{2r-j-\frac{1}{2}}, q^{2r-\frac{1}{2}}; q^{2r-\frac{1}{2}})_\infty}{(q;q)_\infty} \\
=&\frac{(q^j, q^{2r-j-\frac{1}{2}}, q^{2r-\frac{1}{2}}; q^{2r-\frac{1}{2}})_\infty}{(q^{\frac{1}{2}}, q^{\frac{3}{2}}, q^2; q^2)_\infty}.
\end{aligned}$$

The second last identity is due to B. Wang–L. Wang [26, Corollary 4.2]. This completes the proof of Theorem 1.5. ∎

3.2. **Proof of Theorem 1.10.** We invoke (2.7) and (2.8) in Theorem 2.3 for this derivation.

Proof. Let

$$F = \sum_{n_1,\ldots,n_r\geq 0} \frac{q^{\frac{1}{2}\boldsymbol{n}^T AD\boldsymbol{n} + \boldsymbol{b}_{r+1}\boldsymbol{n}}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{n_r}},$$

and

$$F_\sigma = \sum_{\substack{n_1,\ldots,n_r\geq 0\\ n_r\equiv\sigma \pmod 2}} \frac{q^{\frac{1}{2}\boldsymbol{n}^T AD\boldsymbol{n} + \boldsymbol{b}_{r+1}\boldsymbol{n}}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{n_r}},$$

where $\sigma=0,1$. By (3.6),

$$F_0=(-q;q^2)_\infty\sum_{N_1\geq\cdots\geq N_{r-1}\geq 0}\frac{q^{2(N_1^2+\cdots+N_{r-1}^2+N_{r-1})}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^2;q^2)_{N_{r-1}}(q^2;q^4)_{N_{r-1}}}. \tag{3.26}$$

From (3.7), we have

$$F_1=\frac{q^{\frac{r+1}{2}}(-q^2;q^2)_\infty}{(1-q)}\sum_{N_1\geq\cdots\geq N_{r-1}\geq 0}\frac{q^{2(N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{r-2})+4N_{r-1}}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^4;q^4)_{N_{r-1}}(q^3;q^2)_{N_{r-1}}}. \tag{3.27}$$

Let

$$H=\sum_{n_1,\ldots,n_r\geq 0}\frac{q^{\frac{1}{2}\boldsymbol{n}^TAD\boldsymbol{n}+\widetilde{\boldsymbol{b}}_{r+1}\boldsymbol{n}}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{n_r}}.$$

For $\sigma=0,1$, define

$$H_\sigma=\sum_{\substack{n_1,\ldots,n_r\geq 0\\ n_r\equiv\sigma \pmod 2}}\frac{q^{\frac{1}{2}\boldsymbol{n}^TAD\boldsymbol{n}+\widetilde{\boldsymbol{b}}_{r+1}\boldsymbol{n}}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{n_r}}.$$

Then $H=H_0+H_1$.

We have

$$\begin{aligned}
&\frac{1}{2}\boldsymbol{n}^TAD\boldsymbol{n}+\widetilde{\boldsymbol{b}}_{r+1}\boldsymbol{n}\\
&=n_1^2+n_1n_r+n_1+2\Big((n_2+n_3+\cdots+n_{r-1}+\frac{n_r}{2})^2\\
&\quad+(n_3+\cdots+n_{r-1}+\frac{n_r}{2})^2+\cdots(n_{r-1}+\frac{n_r}{2})^2+\left(\frac{n_r}{2}\right)^2\Big)\\
&\quad+2((n_2+n_3+\cdots+n_{r-1}+\frac{n_r}{2})+(n_3+\cdots+n_{r-1}+\frac{n_r}{2})+\cdots\\
&\quad+(n_{r-1}+\frac{n_r}{2})+\frac{n_r}{2})+2\cdot\frac{n_r}{2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&q^{\frac{r-3}{2}}H_0\\
&=q^{\frac{r-3}{2}}\sum_{n_2,\ldots,n_r\geq 0}q^{2((n_2+\cdots+n_r)^2+(n_3+\cdots+n_r)^2+\cdots+(n_{r-1}+n_r)^2+n_r^2)}\\
&\quad\times\frac{q^{2((n_2+\cdots+n_r)+(n_3+\cdots+n_r)+\cdots+(n_{r-1}+n_r)+n_r)+2n_r}}{(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{2n_r}}\sum_{n_1=0}^{\infty}\frac{q^{n_1^2+(2n_r+1)n_1}}{(q^2;q^2)_{n_1}}\\
&\overset{(3.5)}{=}q^{\frac{r-3}{2}}(-q^2;q^2)_\infty\sum_{n_2,\ldots,n_r\geq 0}q^{2((n_2+\cdots+n_r)^2+(n_3+\cdots+n_r)^2+\cdots+(n_{r-1}+n_r)^2+n_r^2)}\\
&\quad\times\frac{q^{2((n_2+\cdots+n_r)+(n_3+\cdots+n_r)+\cdots+(n_{r-1}+n_r)+n_r)+2n_r}}{(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{2n_r}(-q^2;q^2)_{n_r}}
\end{aligned}$$

$$= q^{\frac{r-3}{2}}(-q^2;q^2)_\infty \sum_{n_2,\dots,n_r\geq 0} q^{2((n_2+\cdots+n_r)^2+(n_3+\cdots+n_r)^2+\cdots+(n_{r-1}+n_r)^2+n_r^2)}$$
$$\times \frac{q^{2((n_2+\cdots+n_r)+(n_3+\cdots+n_r)+\cdots+(n_{r-1}+n_r)+n_r)+2n_r}}{(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q^2)_{n_r}(q^4;q^4)_{n_r}}$$
$$= q^{\frac{r-3}{2}}(-q^2;q^2)_\infty$$
$$\times \sum_{N_1\geq N_2\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+N_2^2+\cdots+N_{r-2}^2+N_{r-1}^2)+2(N_1+N_2+\cdots+N_{r-2})+4N_{r-1}}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q;q^2)_{N_{r-1}}(q^4;q^4)_{N_{r-1}}}, \tag{3.28}$$

and

$$q^{\frac{r-3}{2}}H_1$$
$$= q^{\frac{r-3}{2}} \sum_{n_2,\dots,n_r\geq 0} q^{2((n_2+\cdots+n_r+\frac12)^2+(n_3+\cdots+n_r+\frac12)^2+\cdots+(n_{r-1}+n_r+\frac12)^2+(n_r+\frac12)^2)}$$
$$\times \frac{q^{2((n_2+\cdots+n_r+\frac12)+(n_3+\cdots+n_r+\frac12)+\cdots+(n_{r-1}+n_r+\frac12)+(n_r+\frac12))+2(n_r+\frac12)}}{(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q;q)_{2n_r+1}} \sum_{n_1=0}^{\infty} \frac{q^{n_1^2+(2n_r+2)n_1}}{(q^2;q^2)_{n_1}}$$
$$\overset{(3.5)}{=} q^{\frac{r-3}{2}}(-q;q^2)_\infty \sum_{n_2,\dots,n_r\geq 0} q^{2((n_2+\cdots+n_r+\frac12)^2+(n_3+\cdots+n_r+\frac12)^2+\cdots+(n_{r-1}+n_r+\frac12)^2+(n_r+\frac12)^2)}$$
$$\times \frac{q^{2((n_2+\cdots+n_r+\frac12)+(n_3+\cdots+n_r+\frac12)+\cdots+(n_{r-1}+n_r+\frac12)+(n_r+\frac12))+2(n_r+\frac12)}}{(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{r-1}}(q^2;q^2)_{n_r}(q^2;q^4)_{n_r+1}}$$
$$= (-q;q^2)_\infty \sum_{N_1\geq N_2\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+\cdots+N_{r-1}^2)+4(N_1+\cdots N_{r-2})+6N_{r-1}+2r-2}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^2;q^2)_{N_{r-1}}(q^2;q^4)_{N_{r-1}+1}}$$
$$= (-q;q^2)_\infty \sum_{N_1\geq N_2\geq\cdots\geq N_{r-1}\geq 1} \frac{q^{2(N_1^2+\cdots+N_{r-1}^2)+2N_{r-1}}(q^{-2}-q^{2N_{r-1}-2})}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^2;q^2)_{N_{r-1}}(q^2;q^4)_{N_{r-1}}}. \tag{3.29}$$

It follows from (3.26) and (3.29) that

$$F_0 + q^{\frac{r-3}{2}}H_1$$
$$= (-q;q^2)_\infty \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+\cdots+N_{r-1}^2+N_{r-1})}(1+q^{-2}-q^{2N_{r-1}-2})}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^2;q^2)_{N_{r-1}}(q^2;q^4)_{N_{r-1}}}.$$

Substituting $q = q^2$ in (2.7), we have

$$F_0 + q^{\frac{r-3}{2}}H_1 = \frac{(q^{8r+4},q^{8r-8},q^{16r-4};q^{16r-4})_\infty}{(q,q^3,q^4;q^4)_\infty}. \tag{3.30}$$

In view of (3.27) and (3.28), we derive

$$F_1 + q^{\frac{r-3}{2}}H_0$$

$$= q^{\frac{r-3}{2}}(-q^2;q^2)_\infty$$
$$\times \sum_{N_1\geq N_2\geq\cdots\geq N_{r-1}\geq 0} \frac{q^{2(N_1^2+N_2^2+\cdots+N_{r-2}^2+N_{r-1}^2)+2(N_1+N_2+\cdots+N_{r-2})+4N_{r-1}}(1+q^2-q^{2N_{r-1}+1})}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q;q^2)_{N_{r-1}+1}(q^4;q^4)_{N_{r-1}}}.$$

Letting $q\to q^2$ and then $q\to -q$ in (2.8), we have

$$F_1 + q^{\frac{r-3}{2}}H_0 = q^{\frac{r-3}{2}}\frac{(-q^3,q^{4r-4},-q^{4r-1};-q^{4r-1})_\infty}{(q^2,q^2,q^4;q^4)_\infty}. \tag{3.31}$$

Therefore, combining (3.30) and (3.31) yields

$$F + q^{\frac{r-3}{2}}H = \frac{(q^{8r+4},q^{8r-8},q^{16r-4};q^{16r-4})_\infty}{(q,q^3,q^4;q^4)_\infty} + q^{\frac{r-3}{2}}\frac{(-q^3,q^{4r-4},-q^{4r-1};-q^{4r-1})_\infty}{(q^2,q^2,q^4;q^4)_\infty},$$

which is exactly (1.17). ∎

3.3. **Proof of Theorem 1.12.** We prove the two identities stated in the theorem case by case.

(1) **Proof of** (1.22)**.** This follows from (2.5) in Theorem 2.2.

Let $B^\vee D^\vee$ be the symmetric matrix defined by (1.21), and $\boldsymbol{b}_0=(1,1,2,3,\ldots,r-1)^T$. We deduce that

$$\frac{1}{2}\boldsymbol{n}^T B^\vee D^\vee \boldsymbol{n} + \boldsymbol{n}^T\boldsymbol{b}_0$$
$$=\frac{1}{2}n_1^2 + n_1n_r + n_1 + (n_2+\cdots+n_r)^2+\cdots+n_r^2+(n_2+\cdots+n_r)+\cdots+n_r+\frac{1}{2}n_r^2.$$

Then

$$\sum_{\boldsymbol{n}=(n_1,\ldots,n_r)^T\in\mathbb{N}^r} \frac{(-1)^{n_r}q^{\boldsymbol{n}^TB^\vee D^\vee\boldsymbol{n}+2\boldsymbol{n}^T\boldsymbol{b}_0}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{r-1}}(q^4;q^4)_{n_r}}$$
$$= \sum_{n_1,\ldots,n_r\geq 0} \frac{(-1)^{n_r}q^{n_1^2+2n_1n_r+2n_1+2\left((n_2+\cdots+n_r)^2+\cdots+n_r^2+(n_2+\cdots+n_r)+\cdots+n_r\right)+n_r^2}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{r-1}}(q^4;q^4)_{n_r}}$$
$$= \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{(-1)^{N_{r-1}}q^{2\left(N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{r-1}\right)+N_{r-1}^2}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^4;q^4)_{N_{r-1}}}\sum_{n_1=0}^\infty \frac{q^{n_1^2+2n_1N_{r-1}+2n_1}}{(q^2;q^2)_{n_1}}$$
$$\overset{(3.5)}{=} \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{(-1)^{N_{r-1}}q^{2\left(N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{r-1}\right)+N_{r-1}^2}(-q^{2N_{r-1}+3};q^2)_\infty}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^4;q^4)_{N_{r-1}}}$$
$$= (-q^3;q^2)_\infty \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{(-1)^{N_{r-1}}q^{2\left(N_1^2+\cdots+N_{r-1}^2+N_1+\cdots+N_{r-1}\right)+N_{r-1}^2}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(-q^3;q^2)_{N_{r-1}}(q^4;q^4)_{N_{r-1}}}$$
$$\overset{(2.5)}{=} \frac{(-q^3;q^2)_\infty(1+q)(q,q^{4r-4},q^{4r-3};q^{4r-3})_\infty}{(q^2;q^2)_\infty}$$

$$= \frac{(q, q^{4r-4}, q^{4r-3}; q^{4r-3})_\infty}{(q, q^3, q^4; q^4)_\infty},$$

as desired.

(2) **Proof of** (1.23)**.** This derives from (2.6) in Theorem 2.2.

Let $B^\vee D^\vee$ be the symmetric matrix defined by (1.21), and let $\boldsymbol{b}_j = (0, \dots, 0, 1, 2, \dots, r-j-1, r-j-1)^T$. We deduce that

$$\begin{aligned}&\frac{1}{2}\boldsymbol{n}^T B^\vee D^\vee \boldsymbol{n} + \boldsymbol{n}^T \boldsymbol{b}_j \\ =&\frac{1}{2}n_1^2 + n_1 n_r + (n_2 + \cdots + n_r)^2 + \cdots + n_r^2 + (n_{j+1} + \cdots + n_r) + \cdots + n_r + \frac{1}{2}n_r^2 - n_r.\end{aligned}$$

Then

$$\begin{aligned}&\sum_{\boldsymbol{n}=(n_1,\dots,n_r)^T\in\mathbb{N}^r} \frac{(-1)^{n_r} q^{\boldsymbol{n}^T B^\vee D^\vee \boldsymbol{n} + 2\boldsymbol{n}^T \boldsymbol{b}_j}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{r-1}}(q^4;q^4)_{n_r}} \\
&= \sum_{n_1,\dots,n_r\geq 0} \frac{(-1)^{n_r} q^{n_1^2+2n_1n_r+2\left((n_2+\cdots+n_r)^2+\cdots+n_r^2+(n_{j+1}+\cdots+n_r)+\cdots+n_r\right)+n_r^2-2n_r}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{r-1}}(q^4;q^4)_{n_r}} \\
&= \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{(-1)^{N_{r-1}} q^{2\left(N_1^2+\cdots+N_{r-1}+N_j+\cdots+N_{r-1}\right)+N_{r-1}^2-2N_{r-1}}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^4;q^4)_{N_{r-1}}} \sum_{n_1=0}^{\infty} \frac{q^{n_1^2+2n_1N_{r-1}}}{(q^2;q^2)_{n_1}} \\
&\overset{(3.5)}{=} \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{(-1)^{N_{r-1}} q^{2\left(N_1^2+\cdots+N_{r-1}^2+N_j+\cdots+N_{r-1}\right)+N_{r-1}^2-2N_{r-1}}(-q^{2N_{r-1}+1};q^2)_\infty}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(q^4;q^4)_{N_{r-1}}} \\
&= (-q;q^2)_\infty \sum_{N_1\geq\cdots\geq N_{r-1}\geq 0} \frac{(-1)^{N_{r-1}} q^{2\left(N_1^2+\cdots+N_{r-1}^2+N_j+\cdots+N_{r-1}\right)+N_{r-1}^2-2N_{r-1}}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{r-2}-N_{r-1}}(-q;q^2)_{N_{r-1}}(q^4;q^4)_{N_{r-1}}} \\
&\overset{(2.6)}{=} \frac{(-q;q^2)_\infty (q^{2j}, q^{4r-3-2j}, q^{4r-3}; q^{4r-3})_\infty}{(q^2;q^2)_\infty} \\
&= \frac{(q^{2j}, q^{4r-3-2j}, q^{4r-3}; q^{4r-3})_\infty}{(q, q^3, q^4; q^4)_\infty},\end{aligned}$$

which is (1.23). This finishes the proof of Theorem 1.12. ∎

4. Proofs of Theorems 1.1, 1.2 and 1.3

Armed with Theorem 1.5, we are now in a position to establish the modularity of the generalized Nahm sums specified in Theorems 1.1, 1.2 and 1.3. Let us first review some knowledge on modular functions. It turns out that such families of the generalized Nahm sums are certain modular functions for some congruence subgroup $\Gamma_1(N)$.

For a positive integer N, the congruence subgroups $\Gamma_0(N)$ and $\Gamma_1(N)$ are defined as

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \colon c \equiv 0 \bmod N \right\},$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \colon a \equiv d \equiv 1 \bmod N \right\},$$

where Γ is the full modular group given by

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \text{ and } ad - bc = 1 \right\}.$$

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ act on $\tau \in \mathbb{C}$ by the linear fractional transformation

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}, \qquad \text{and} \qquad \gamma\infty = \lim_{\tau \to \infty} \gamma\tau.$$

Definition 4.1. ([7, Chapter 3]) *Let $\mathbb{H} = \{\tau \in \mathbb{C} \colon \operatorname{Im}(\tau) > 0\}$. A meromorphic function $f \colon \mathbb{H} \to \mathbb{C}$ is called a modular function of weight $k \in \mathbb{N}^+$ for $\Gamma_1(N)$, if it satisfies the following two conditions:*

(1) *for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$, $f(\gamma\tau) = (c\tau + d)^k f(\tau)$;*

(2) *for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $(c\tau + d)^{-k} f(\gamma\tau)$ has a Fourier expansion of the form*

$$(c\tau + d)^{-k} f(\gamma\tau) = \sum_{n=n_\gamma}^{\infty} a(n) q_{w_\gamma}^n,$$

where $a(n_\gamma) \neq 0$, $q_{w_\gamma} = e^{2\pi i\tau/w_\gamma}$, and w_γ is the minimal positive integer h such that

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \gamma^{-1}\Gamma_1(N)\gamma.$$

A modular function of weight 0 for $\Gamma_1(N)$ is referred to a modular function for $\Gamma_1(N)$.

As shown in Theorem 1.5, we find that the generalized Nahm sums specified in Theorems 1.1, 1.2 and 1.3 can be expressed as generalized eta-quotients or sums of generalized eta-quotients. Generalized eta-quotients can be viewed as the quotients of generalized Dedekind eta-functions.

For a positive integer δ and a residue class $g \pmod{\delta}$, the generalized Dedekind eta-function $\eta_{\delta,g}(\tau)$ is defined by

$$\eta_{\delta,g}(\tau) = q^{\frac{\delta}{2}P_2\left(\frac{g}{\delta}\right)} \prod_{\substack{n>0 \\ n\equiv g \pmod{\delta}}} (1-q^n) \prod_{\substack{n>0 \\ n\equiv -g \pmod{\delta}}} (1-q^n),$$

where $q = e^{2\pi i\tau}$ and

$$P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$$

is the second Bernoulli function and $\{t\}$ is the fractional part of t; see, for example, [19, 20].

The following criterion established by Robins [19, Theorem 3] can be used to determine the modularity of generalized eta-quotients. In particular, the generalized eta-quotients linked to the generalized Nahm sums outlined in Theorems 1.1, 1.2, and 1.3 also fall into this category.

Lemma 4.2. ([19, Theorem 3]) *For the generalized eta-quotient $f(\tau)$ of the following form:*

$$f(\tau) = \prod_{\substack{\delta|N \\ 0\le g<\delta}} \eta_{\delta,g}^{r_{\delta,g}}(\tau), \tag{4.1}$$

where

$$r_{\delta,g} \in \begin{cases} \frac{1}{2}\mathbb{Z}, & \text{if } g = 0 \text{ or } \frac{\delta}{2}, \\ \mathbb{Z}, & \text{otherwise}, \end{cases}$$

we assume that $f(\tau)$ satisfies the following three conditions:

$$(1)\ w(f) = \sum_{\delta|N} r_{\delta,0} = 0; \quad (2)\ \mathrm{Ord}_\infty(f) = \sum_{\substack{\delta|N \\ 0\le g<\delta}} \delta P_2\left(\frac{g}{\delta}\right) r_{\delta,g} = \frac{m_1}{n_1};$$

$$(3)\ \mathrm{Ord}_0(f) = \sum_{\substack{\delta|N \\ 0\le g<\delta}} \frac{N}{\delta} P_2(0) r_{\delta,g} = \frac{m_2}{n_2},$$

where $m_j, n_j \in \mathbb{Z}$, $n_j > 0, j = 1, 2$. Let t and N_0 be the least positive integers such that both tm_1/n_1 and N_0m_2/n_2 are even integers. Then $f(t\tau)$ is a modular function for $\Gamma_1(tN_0N)$.

We now prove Theorems 1.1, 1.2 and 1.3 with the aid of Theorem 1.5 and Lemma 4.2.

Proof of Theorem 1.1. Let $(A, \boldsymbol{b}_j, c_j)$ be the triple of index $\boldsymbol{d} = (2, \ldots, 2, 1)$ given in Theorem 1.1. It follows from Theorem 1.5 (1) and (2) that, for $0 \le j \le r$,

$$\widetilde{f}_{A,\boldsymbol{b}_j,c_j,\boldsymbol{d}}(\tau) = \widetilde{f}_{A,\boldsymbol{b}_j,c_j,\boldsymbol{d}}(q)$$

$$= \frac{\eta_{16r-4,4k_j}(\tau)\eta^{1/2}_{16r-4,0}(\tau)}{\eta_{4,1}(\tau)\eta^{1/2}_{4,0}(\tau)} + \frac{\eta_{8r-2,2k_j}(\tau)\eta_{16r-4,8r-4k_j-2}(\tau)\eta^{3/2}_{8r-2,0}(\tau)}{\eta_{8r-2,4r-2k_j-1}(\tau)\eta_{4,2}(\tau)\eta^{1/2}_{4,0}(\tau)\eta^{1/2}_{4r-1,0}(\tau)\eta^{1/2}_{16r-4,0}(\tau)},$$

where $k_0 = 2r-1$ and $k_j = j$ for $1 \le j \le r$.

Let

$$f_1(\tau) = \frac{\eta_{16r-4,4k_j}(\tau)\eta^{1/2}_{16r-4,0}(\tau)}{\eta_{4,1}(\tau)\eta^{1/2}_{4,0}(\tau)},$$

$$f_2(\tau) = \frac{\eta_{8r-2,2k_j}(\tau)\eta_{16r-4,8r-4k_j-2}(\tau)\eta^{3/2}_{8r-2,0}(\tau)}{\eta_{8r-2,4r-2k_j-1}(\tau)\eta_{4,2}(\tau)\eta^{1/2}_{4,0}(\tau)\eta^{1/2}_{4r-1,0}(\tau)\eta^{1/2}_{16r-4,0}(\tau)}.$$

Then $\widetilde{f}_{A,\boldsymbol{b}_j,c_j,\boldsymbol{d}}(\tau) = f_1(\tau) + f_2(\tau)$. Take $N = 16r-4$. Then we have

$$w(f_1) = \frac{1}{2} - \frac{1}{2} = 0, \quad \mathrm{Ord}_0(f_1) = -r + \frac{1}{2},$$
$$\mathrm{Ord}_\infty(f_1) = \frac{64r^2 - (64k_j+36)r + 16k_j^2 + 16k_j + 5}{16r-4},$$

and

$$w(f_2) = \frac{3}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = 0,$$
$$\mathrm{Ord}_\infty(f_2) = \frac{(4r-4k_j-1)^2}{16r-4}, \quad \mathrm{Ord}_0(f_2) = -r + \frac{1}{2}.$$

Let $t = 32r-8$ and $N_0 = 4$. Then

$$t \cdot \mathrm{Ord}_\infty(f_1) = 2(64r^2 - (64k_j+36)r + 16k_j^2 + 16k_j + 5),$$
$$t \cdot \mathrm{Ord}_\infty(f_2) = 2(4r-4k_j-1)^2$$

and

$$N_0 \cdot \mathrm{Ord}_0(f_1) = N_0 \cdot \mathrm{Ord}_0(f_2) = 2(-2r+1)$$

are even. By Lemma 4.2, we know that $f_1((32r-8)\tau)$ and $f_2((32r-8)\tau)$ are modular functions for $\Gamma_1(128(4r-1)^2)$. Therefore, the function $\widetilde{f}_{A,\boldsymbol{b}_j,c_j,\boldsymbol{d}}((32r-8)\tau)$ (i.e., $\widetilde{f}_{A,\boldsymbol{b}_j,c_j,\boldsymbol{d}}(q^{32r-8})$) is a modular function for $\Gamma_1(128(4r-1)^2)$. ∎

Proof of Theorem 1.2. Let $(B, \boldsymbol{b}_j, c_j)$ be the triple of index $\boldsymbol{d} = (2, \ldots, 2, 1)$ specified in Theorem 1.2. It follows from Theorem 1.5 (3) and (4) that, for $0 \le j \le r$, we have

$$\widetilde{f}_{B,\boldsymbol{b}_j,c_j,\boldsymbol{d}}(\tau) = \widetilde{f}_{B,\boldsymbol{b}_j,c_j,\boldsymbol{d}}(q)$$
$$= \frac{\eta_{16r-12,4\tilde{k}_j}(\tau)\eta^{1/2}_{16r-12,0}(\tau)}{\eta_{4,1}(\tau)\eta^{1/2}_{4,0}(\tau)} + \frac{\eta_{8r-6,2\tilde{k}_j}(\tau)\eta_{16r-12,8r-4\tilde{k}_j-6}(\tau)\eta^{3/2}_{8r-6,0}(\tau)}{\eta_{8r-6,4r-2\tilde{k}_j-3}(\tau)\eta_{4,2}(\tau)\eta^{1/2}_{4,0}(\tau)\eta^{1/2}_{4r-3,0}(\tau)\eta^{1/2}_{16r-12,0}(\tau)},$$

where $\tilde{k}_0 = 2r-2$ and $\tilde{k}_j = j$ and $1 \le j \le r$.

Let

$$g_1(\tau) = \frac{\eta_{16r-12,4\tilde{k}_j}(\tau)\eta^{1/2}_{16r-12,0}(\tau)}{\eta_{4,1}(\tau)\eta^{1/2}_{4,0}(\tau)},$$

$$g_2(\tau) = \frac{\eta_{8r-6,2\tilde{k}_j}(\tau)\eta_{16r-12,8r-4\tilde{k}_j-6}(\tau)\eta^{3/2}_{8r-6,0}(\tau)}{\eta_{8r-6,4r-2\tilde{k}_j-3}(\tau)\eta_{4,2}(\tau)\eta^{1/2}_{4,0}(\tau)\eta^{1/2}_{4r-3,0}(\tau)\eta^{1/2}_{16r-12,0}(\tau)}.$$

Then $\widetilde{f}_{B,\boldsymbol{b}_j,c_j,\boldsymbol{d}}(\tau) = g_1(\tau) + g_2(\tau)$. Set $\tilde{N} = 16r - 12$, we have

$$w(g_1) = \frac{1}{2} - \frac{1}{2} = 0, \quad \mathrm{Ord}_0(g_1) = 1 - r,$$

$$\mathrm{Ord}_\infty(g_1) = \frac{64r^2 - (64k_j + 100)r + 16\tilde{k}_j^2 + 48\tilde{k}_j + 39}{4(4r-3)},$$

and

$$w(g_2) = \frac{3}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = 0, \quad \mathrm{Ord}_0(g_2) = 1 - r,$$

$$\mathrm{Ord}_\infty(g_2) = \frac{(4r - 4\tilde{k}_j - 3)^2}{4(4r-3)}.$$

Let $\tilde{t} = 32r - 24$ and $\tilde{N}_0 = 2$. Then

$$\tilde{t} \cdot \mathrm{Ord}_\infty(g_1) = 2(64r^2 - (64\tilde{k}_j + 100)r + 16\tilde{k}_j^2 + 48\tilde{k}_j + 39),$$
$$\tilde{t} \cdot \mathrm{Ord}_\infty(g_2) = 2(4r - 4\tilde{k}_j - 3)^2$$

and

$$\tilde{N}_0 \cdot \mathrm{Ord}_0(g_1) = \tilde{N}_0 \cdot \mathrm{Ord}_0(g_2) = 2(1-r)$$

are even. By Lemma 4.2, we have $g_1((32r-24)\tau)$ and $g_2((32r-24)\tau)$ are modular functions for $\Gamma_1(64(4r-3)^2)$. Hence, $\widetilde{f}_{B,\boldsymbol{b}_j,c_j,\boldsymbol{d}}((32r-24)\tau)$ (i.e., $\widetilde{f}_{B,\boldsymbol{b}_j,c_j,\boldsymbol{d}}(q^{32r-24})$) is a modular function for $\Gamma_1(64(4r-3)^2)$. This completes the proof. ∎

Proof of Theorem 1.3. Let $(A^\vee, \boldsymbol{b}_j, c_j)$ $(0 \le j \le r)$ be the triple of index $\boldsymbol{d}^\vee = (1, \ldots, 1, 2)$ specified in Theorem 1.3. From Theorem 1.5 (5) and (6), we can see that, for $0 \le j \le r$,

$$\widetilde{f}_{A^\vee,\boldsymbol{b}_j,c_j,\boldsymbol{d}^\vee}(\tau) = \widetilde{f}_{A^\vee,\boldsymbol{b}_j,c_j,\boldsymbol{d}^\vee}(q) = \frac{\eta_{2r-\frac{1}{2},k_j}(\tau)\eta^{1/2}_{2r-\frac{1}{2},0}(\tau)}{\eta_{2,\frac{1}{2}}(\tau)\eta^{1/2}_{2,0}(\tau)},$$

where $k_0 = \frac{1}{2}$ and $k_j = j$ for $1 \le j \le r$.

Take $N = 4r - 1$. Then we have

$$w(\widetilde{f}_{A^\vee,\boldsymbol{b}_j,c_j,\boldsymbol{d}^\vee}(\tau)) = 0, \quad \mathrm{Ord}_0(\widetilde{f}_{A^\vee,\boldsymbol{b}_j,c_j,\boldsymbol{d}^\vee}(\tau)) = \frac{5-4r}{8},$$

$$\mathrm{Ord}_\infty(\widetilde{f}_{A^\vee,\boldsymbol{b}_j,c_j,\boldsymbol{d}^\vee}(\tau)) = \frac{8r^2-2(8k_j+3)r+8{k_j}^2+4k_j+1}{16r-4}.$$

Let $t = 32r-8$ and $N_0 = 16$. Then both

$$t\cdot \mathrm{Ord}_\infty(\widetilde{f}_{A^\vee,\boldsymbol{b}_j,c_j,\boldsymbol{d}^\vee}(\tau)) \text{ and } N_0\cdot \mathrm{Ord}_0(\widetilde{f}_{A^\vee,\boldsymbol{b}_j,c_j,\boldsymbol{d}^\vee}(\tau))$$

are even. It follows from Lemma 4.2 that $\widetilde{f}_{A^\vee,\boldsymbol{b}_j,c_j,\boldsymbol{d}^\vee}((32r-8)\tau)$ (that is, $\widetilde{f}_{A^\vee,\boldsymbol{b}_j,c_j,\boldsymbol{d}^\vee}(q^{32r-8})$) is a modular function for $\Gamma_1(128(4r-1)^2)$. ∎

5. Modularity for vector-valued functions

In this section, we will prove Theorem 1.6 and Theorem 1.7. We first recall some necessary definitions and background in Subsection 5.1. Then based on Theorems 1.1, 1.2 and 1.3, we construct four vector-valued functions, divide them into two "Langlands dual" pairs, and derive modular transformation formulas for each pair. Using these transformation formulas, we prove Theorem 1.6 and Theorem 1.7 in Subsection 5.2 and Subsection 5.3, respectively.

5.1. **Preliminary.** Let Γ' be a discrete subgroup of

$$PSL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d\in\mathbb{R},\ \text{and } ad-bc>0 \right\}$$

with $\rho\colon \Gamma'\to GL_n(\mathbb{C})$ a representation of Γ'. A column vector of meromorphic functions $\boldsymbol{F}(\tau) = (f_1(\tau), f_2(\tau),\ldots,f_n(\tau))^T$ on the upper half-plane is called a vector-valued automorphic form of multiplier ρ and integral weight k with respect to Γ', if it satisfies for $\gamma\in\Gamma'$,

$$\boldsymbol{F}(\gamma\tau) = \rho(\gamma)(c\tau+d)^k\boldsymbol{F}(\tau).$$

See [21].

Furthermore, suppose Γ' is a finite index subgroup of Γ and let $\boldsymbol{F}(\tau) = (f_1(\tau), f_2(\tau), \ldots, f_n(\tau))^T$ be a vector-valued automorphic form of multiplier ρ and integral weight k for Γ'. If, in addition, for all $\gamma\in\Gamma$ and each component function $f_i(\tau)$, there exists a positive integer $N_{\gamma,i}$ such that

$$f_i(\gamma\tau) = (c\tau+d)^k\sum_n a_n e^{2\pi i n\tau/N_{\gamma,i}}$$

with $a_n = 0$ for $n\ll 0$, then $\boldsymbol{F}(\tau)$ is called a vector-valued modular function of integral weight k associated with ρ for Γ'; see [12].

A vector-valued automorphic form (modular function) of weight 0 associated with ρ with respect to Γ' is referred to a vector-valued automorphic form (modular function) associated with ρ for Γ'.

To prove Theorem 1.6 and Theorem 1.7, we first recall some properties of Weber's modular functions and theta series. The Dedekind eta function is defined as follows:

$$\eta(\tau) = q^{1/24}(q;q)_\infty,$$

and Weber's modular functions are defined as

$$\mathfrak{f}(\tau) = q^{-1/48}(-q^{1/2};q)_\infty,\ \mathfrak{f}_1(\tau) = q^{-1/48}(q^{1/2};q)_\infty,\ \mathfrak{f}_2(\tau) = q^{1/24}(-q;q)_\infty,$$

where $q = e^{2\pi i\tau}$ and $\tau \in \mathbb{H}$. They have the following properties:

$$\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau), \quad \eta(\tau+1) = e^{\pi i/12}\eta(\tau), \tag{5.1}$$

$$\mathfrak{f}(-1/\tau) = \mathfrak{f}(\tau), \quad \mathfrak{f}_2(-1/\tau) = \frac{1}{\sqrt{2}}\mathfrak{f}_1(\tau), \quad \mathfrak{f}_1(-1/\tau) = \sqrt{2}\mathfrak{f}_2(\tau), \tag{5.2}$$

$$\mathfrak{f}(\tau+1) = e^{-\pi i/24}\mathfrak{f}_1(\tau), \quad \mathfrak{f}_1(\tau+1) = e^{-\pi i/24}\mathfrak{f}(\tau), \quad \mathfrak{f}_2(\tau+1) = e^{\pi i/12}\mathfrak{f}_2(\tau). \tag{5.3}$$

For $m \in \mathbb{Q}_{>0}$ and $j \in \mathbb{Q}$, define the theta series

$$h_{j,m}(\tau) = \sum_{k\in\mathbb{Z}} q^{m\left(k+\frac{j}{2m}\right)^2}, \quad g_{j,m}(\tau) = \sum_{k\in\mathbb{Z}} (-1)^k q^{m\left(k+\frac{j}{2m}\right)^2}.$$

It follows from the Jacobi triple product identity (2.11) that

$$h_{j,m}(\tau) = q^{\frac{j^2}{4m}}\left(-q^{m-j}, -q^{m+j}, q^{2m}; q^{2m}\right)_\infty, \tag{5.4}$$

$$g_{j,m}(\tau) = q^{\frac{j^2}{4m}}\left(q^{m+j}, q^{m-j}, q^{2m}; q^{2m}\right)_\infty. \tag{5.5}$$

Moreover, $h_{j,m}(\tau)$ and $g_{j,m}(\tau)$ satisfy the following properties:

$$h_{j,m}(\tau) = h_{-j,m}(\tau) = h_{2m+j,m}(\tau), \quad g_{j,m}(\tau) = g_{-j,m}(\tau) = -g_{2m+j,m}(\tau), \tag{5.6}$$

$$h_{j,m}(\tau) = h_{2j,4m}(\tau) + h_{4m-2j,4m}(\tau), \tag{5.7}$$

$$g_{j,m}(\tau) = h_{2j,4m}(\tau) - h_{4m-2j,4m}(\tau), \tag{5.8}$$

$$h_{j,m}(2\tau) = h_{2j,2m}(\tau), \quad g_{j,m}(2\tau) = g_{2j,2m}(\tau). \tag{5.9}$$

Lemma 5.1. ([25, Theorem 4.5]) *For $j \in \mathbb{Z}$ and $m \in \frac{1}{2}\mathbb{N}$,*

$$g_{j,m}\left(-\frac{1}{\tau}\right) = \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m}} \sum_{\substack{0\le k\le 4m-1\\ k\ odd}} e^{\frac{\pi ijk}{2m}} h_{\frac{k}{2},m}(\tau).$$

Lemma 5.2. ([27, Lemma 2.13]) *For any integer m and an odd integer $1 \le j \le m$,*

$$g_{j,m}\left(-\frac{\tau+1}{4\tau}\right) = \sqrt{\frac{-\tau}{m}}\epsilon_m \sum_{\substack{1\le\ell\le m-1\\ \ell\ odd}} \left(e^{\pi i\frac{1-(j+\ell)-(j+\ell-2)^2\delta_m}{2m}} + e^{\pi i\frac{1-(j-\ell)-(j-\ell-2)^2\delta_m}{2m}}\right) g_{\ell,m}\left(\frac{\tau+1}{4}\right),$$

where

$$\epsilon_m = \begin{cases} 1, & \textit{if } m \equiv 1 \pmod 4, \\ i, & \textit{if } m \equiv 3 \pmod 4, \end{cases}$$

and δ_m *is any integer satisfying* $4\delta_m \equiv 1 \pmod m$*.*

Remark 5.3. *Notice that we can choose* $\delta_{4r-1} = r$ *and* $\delta_{4r-3} = 3r - 2$*.*

We also need the following lemma.

Lemma 5.4. *Let* N *be an odd integer,* $m = \frac{N-1}{2}$*,* $S = (s_{j,k})_{m\times m}$*, where* $s_{j,k} = \sqrt{\frac{2}{N}} \cos \frac{(2j-1)(2k-1)\pi}{2N}$*. Then* $2S^2 = E_m$*, where* E_m *is the* $m \times m$ *identity matrix.*

Proof. To prove $S^2 = \frac{1}{2}E_m$, let $a_{j,k}$ denote the (j,k)-entry of S^2. By matrix multiplication,

$$\begin{aligned} a_{j,k} &= \sum_{\ell=1}^{m} s_{j,\ell} s_{\ell,k} \\ &= \frac{2}{N} \sum_{\ell=1}^{m} \cos \frac{(2j-1)(2\ell-1)\pi}{2N} \cos \frac{(2k-1)(2\ell-1)\pi}{2N} \\ &= \frac{1}{N} \left(\sum_{\ell=1}^{m} \cos \frac{(j+k-1)(2\ell-1)\pi}{N} + \sum_{\ell=1}^{m} \cos \frac{(j-k)(2\ell-1)\pi}{N} \right). \end{aligned}$$

Next we show that

$$C(t) := \sum_{\ell=1}^{m} \cos \frac{t(2\ell-1)\pi}{N} = \begin{cases} m, & \text{if } t = 0, \\ \frac{(-1)^{t+1}}{2}, & \text{if } t \neq 0 \text{ and } \sin \frac{t\pi}{N} \neq 0. \end{cases}$$

For $t = 0$, it is easy to see that $C(t) = m$. For t with $\sin \frac{t\pi}{N} \neq 0$,

$$\begin{aligned} C(t) &= \mathrm{Re} \sum_{\ell=1}^{m} e^{\frac{it(2\ell-1)\pi}{N}} = \mathrm{Re} \left(e^{ti\pi/N} \frac{1 - e^{2mti\pi/N}}{1 - e^{2ti\pi/N}} \right) \\ &= \mathrm{Re} \left(\frac{1 - (-1)^t e^{-ti\pi/N}}{-2i \sin \frac{t\pi}{N}} \right) \\ &= \mathrm{Re} \left(\frac{1 - (-1)^t (\cos \frac{t\pi}{N} - i \sin \frac{t\pi}{N})}{-2i \sin \frac{t\pi}{N}} \right) = \frac{(-1)^{t+1}}{2}. \end{aligned}$$

Thus, for $1 \le j = k \le m$,

$$a_{j,j} = \frac{1}{N}(C(2j-1) + C(0)) = \frac{1}{N}\left(\frac{1}{2} + m\right) = \frac{1}{2},$$

and for $1 \le j, k \le m$ with $j \neq k$, we have $1 \le j + k - 1 \le 2m - 1 < N$ and $1 \le |j - k| \le m - 1 < N$. So

$$a_{j,k} = \frac{1}{N}(C(j+k-1) + C(j-k)) = \frac{1}{N}\left(\frac{(-1)^{j+k}}{2} + \frac{(-1)^{j-k+1}}{2}\right) = 0.$$

Therefore, $S^2 = \frac{1}{2}E_m$. ∎

5.2. **Proof of Theorem 1.6.** Recall that

$$\boldsymbol{g}_{2r-1}(\tau) = (g_1(\tau), g_2(\tau), \dots, g_{2r-1}(\tau))^T, \quad \boldsymbol{g}^\vee_{2r-1}(\tau) = (g^\vee_1(\tau), g^\vee_2(\tau), \dots, g^\vee_{2r-1}(\tau))^T,$$

where $g_j(\tau)$ and $g^\vee_j(\tau)$ are given in (1.13) and (1.14).

We obtain the following modular transformation formula between $\boldsymbol{g}_{2r-1}(\tau)$ and $\boldsymbol{g}^\vee_{2r-1}(\tau)$, which is a "Langlands dual" pair similar to that of [15].

Theorem 5.5. *For $r \geq 2$,*

$$\boldsymbol{g}_{2r-1}\Big(-\frac{1}{2\tau}\Big) = 2S\,\boldsymbol{g}^\vee_{2r-1}(\tau), \quad \boldsymbol{g}^\vee_{2r-1}\Big(-\frac{1}{2\tau}\Big) = S\,\boldsymbol{g}_{2r-1}(\tau),$$

where $S = (s_{j,k})_{(2r-1)\times(2r-1)}$ and $s_{j,k} = \sqrt{\frac{2}{4r-1}}\cos\frac{(2j-1)(2k-1)\pi}{2(4r-1)}$.

Proof. By Lemma 5.4, it suffices to prove

$$\boldsymbol{g}_{2r-1}\Big(-\frac{1}{2\tau}\Big) = 2S\,\boldsymbol{g}^\vee_{2r-1}(\tau). \tag{5.10}$$

Let

$$V_j(\tau) = q^{\frac{16j^2-16j-4r+5}{32r-8}}\frac{(q^{8r-4j}, q^{8r+4j-4}, q^{16r-4}; q^{16r-4})_\infty}{(q, q^3, q^4; q^4)_\infty},$$

$$V^*_j(\tau) = q^{\frac{(4r-4j+1)^2}{32r-8}}\frac{(-q^{2j-1}, q^{4r-2j}, -q^{4r-1}; -q^{4r-1})_\infty}{(q^2, q^2, q^4; q^4)_\infty},$$

$$U_j(2\tau) = q^{\frac{2j^2-2j-2r+1}{16r-4}}\frac{(q^{2r-j}, q^{2r+j-1}, q^{4r-1}; q^{4r-1})_\infty}{(q, q^3, q^4; q^4)_\infty}, \tag{5.11}$$

$$U^*_j(2\tau) = (-1)^{\frac{j(j-1)}{2}}q^{\frac{2j^2-2j-2r+1}{16r-4}}\frac{((-q)^{2r-j}, (-q)^{2r+j-1}, -q^{4r-1}; -q^{4r-1})_\infty}{(-q, -q^3, q^4; q^4)_\infty}. \tag{5.12}$$

Then for $1 \leq j \leq 2r-1$,

$$g_j(\tau) = V_j(\tau) + V^*_j(\tau), \quad g^\vee_j(2\tau) = \frac{1}{2}(U_j(2\tau) + U^*_j(2\tau)). \tag{5.13}$$

Let

$$\boldsymbol{V}_{2r-1}(\tau) = (V_1(\tau), V_2(\tau), \dots, V_{2r-1}(\tau))^T,$$

$$\boldsymbol{V}^*_{2r-1}(\tau) = (V^*_1(\tau), V^*_2(\tau), \dots, V^*_{2r-1}(\tau))^T,$$

$$\boldsymbol{U}_{2r-1}(\tau) = (U_1(\tau), U_2(\tau), \dots, U_{2r-1}(\tau))^T$$

and

$$\boldsymbol{U}^*_{2r-1}(\tau) = (U^*_1(\tau), U^*_2(\tau), \dots, U^*_{2r-1}(\tau))^T.$$

We have

$$\boldsymbol{g}_{2r-1}(\tau)=\boldsymbol{V}_{2r-1}(\tau)+\boldsymbol{V}^*_{2r-1}(\tau),\quad \boldsymbol{g}^{\vee}_{2r-1}(2\tau)=\frac{1}{2}(\boldsymbol{U}_{2r-1}(2\tau)+\boldsymbol{U}^*_{2r-1}(2\tau)).$$

Hence, in order to prove (5.10), we only need to prove

$$\boldsymbol{V}_{2r-1}\left(-\frac{1}{2\tau}\right)=S\,\boldsymbol{U}_{2r-1}(\tau), \tag{5.14}$$

and

$$\boldsymbol{V}^*_{2r-1}\left(-\frac{1}{2\tau}\right)=S\,\boldsymbol{U}^*_{2r-1}(\tau). \tag{5.15}$$

Next we prove (5.14) and (5.15), respectively.

Proof of (5.14)**.** In view of (5.5) and the following identity

$$\frac{1}{(q,q^3,q^4;q^4)_\infty}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty},$$

we see that

$$\begin{aligned}V_j(\tau)&=q^{\frac{16j^2-16j-4r+5}{32r-8}}\frac{(-q;q^2)_\infty(q^{8r-4j},q^{8r+4j-4},q^{16r-4};q^{16r-4})_\infty}{(q^2;q^2)_\infty}\\&=\frac{\mathfrak{f}(2\tau)}{\eta(2\tau)}g_{2j-1,4r-1}(2\tau),\end{aligned}$$

and

$$U_j(2\tau)=\frac{\mathfrak{f}(2\tau)}{\eta(2\tau)}g_{2j-1,4r-1}\left(\frac{\tau}{2}\right). \tag{5.16}$$

Then

$$V_j\left(-\frac{1}{4\tau}\right)=\frac{\mathfrak{f}(-\frac{1}{2\tau})}{\eta(-\frac{1}{2\tau})}g_{2j-1,4r-1}\left(-\frac{1}{2\tau}\right).$$

By means of (5.1), (5.2) and Lemma 5.1, we have

$$\begin{aligned}V_j\left(-\frac{1}{4\tau}\right)&=\frac{1}{\sqrt{8r-2}}\frac{\mathfrak{f}(2\tau)}{\eta(2\tau)}\sum_{\substack{0\le k\le 16r-5\\ k\text{ odd}}}e^{\frac{\pi i(2j-1)k}{2(4r-1)}}h_{\frac{k}{2},4r-1}(2\tau)\\&=\frac{1}{\sqrt{8r-2}}\frac{\mathfrak{f}(2\tau)}{\eta(2\tau)}\sum_{k=0}^{8r-3}e^{\frac{\pi i(2j-1)(2k+1)}{2(4r-1)}}h_{k+\frac{1}{2},4r-1}(2\tau).\end{aligned}$$

In view of (5.6), we deduce that

$$\begin{aligned}&V_j\left(-\frac{1}{4\tau}\right)\\&=\frac{1}{\sqrt{8r-2}}\frac{\mathfrak{f}(2\tau)}{\eta(2\tau)}\sum_{k=0}^{4r-2}\left(e^{\frac{\pi i(2j-1)(2k+1)}{2(4r-1)}}h_{k+\frac{1}{2},4r-1}(2\tau)+e^{\frac{\pi i(2j-1)(-2k-1)}{2(4r-1)}}h_{k+\frac{1}{2},4r-1}(2\tau)\right)\end{aligned}$$

$$= \sqrt{\frac{2}{4r-1}}\frac{\mathfrak{f}(2\tau)}{\eta(2\tau)}\sum_{k=0}^{4r-2}\cos\frac{\pi(2j-1)(2k+1)}{2(4r-1)}h_{k+\frac{1}{2},4r-1}(2\tau)$$

$$= \sqrt{\frac{2}{4r-1}}\frac{\mathfrak{f}(2\tau)}{\eta(2\tau)}\sum_{k=0}^{2r-2}\cos\frac{\pi(2j-1)(2k+1)}{2(4r-1)}(h_{k+\frac{1}{2},4r-1}(2\tau)-h_{4r-k-\frac{3}{2},4r-1}(2\tau)),$$

which, together with (5.9), yields that

$$V_j\Big(-\frac{1}{4\tau}\Big)$$
$$= \sqrt{\frac{2}{4r-1}}\frac{\mathfrak{f}(2\tau)}{\eta(2\tau)}\sum_{k=0}^{2r-2}\cos\frac{\pi(2j-1)(2k+1)}{2(4r-1)}\bigg(h_{4k+2,16r-4}\Big(\frac{\tau}{2}\Big)-h_{16r-4k-6,16r-4}\Big(\frac{\tau}{2}\Big)\bigg).$$

From (5.8), we have

$$V_j\Big(-\frac{1}{4\tau}\Big) = \sqrt{\frac{2}{4r-1}}\frac{\mathfrak{f}(2\tau)}{\eta(2\tau)}\sum_{k=0}^{2r-2}\cos\frac{\pi(2j-1)(2k+1)}{2(4r-1)}g_{2k+1,4r-1}\Big(\frac{\tau}{2}\Big)$$
$$= \sqrt{\frac{2}{4r-1}}\sum_{k=1}^{2r-1}\cos\frac{\pi(2j-1)(2k-1)}{2(4r-1)}\frac{\mathfrak{f}(2\tau)}{\eta(2\tau)}g_{2k-1,4r-1}\Big(\frac{\tau}{2}\Big).$$

Hence,

$$V_j\Big(-\frac{1}{2\tau}\Big) = \sqrt{\frac{2}{4r-1}}\sum_{k=1}^{2r-1}\cos\frac{\pi(2j-1)(2k-1)}{2(4r-1)}\frac{\mathfrak{f}(\tau)}{\eta(\tau)}g_{2k-1,4r-1}\Big(\frac{\tau}{4}\Big)$$
$$= \sqrt{\frac{2}{4r-1}}\sum_{k=1}^{2r-1}\cos\frac{\pi(2j-1)(2k-1)}{2(4r-1)}U_k(\tau)\ \text{(by (5.16))},$$

which gives (5.14).

Proof of (5.15)**.** Since

$$\frac{1}{(q^2,q^2,q^4;q^4)} = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty},$$

we have

$$V_j^*(\tau+\frac{1}{2}) = e^{\pi i\frac{(4r-4j+1)^2}{32r-8}}q^{\frac{(4r-4j+1)^2}{32r-8}}\frac{(-q^2;q^2)_\infty(q^{2j-1},q^{4r-2j},q^{4r-1};q^{4r-1})_\infty}{(q^2;q^2)_\infty}$$
$$= \begin{cases} e^{\pi i\frac{(4r-4j+1)^2}{32r-8}}\frac{\mathfrak{f}_2(2\tau)}{\eta(2\tau)}g_{4r-4j+1,4r-1}\big(\frac{\tau}{2}\big), & \text{if } 1\le j\le r,\\ e^{\pi i\frac{(4r-4j+1)^2}{32r-8}}\frac{\mathfrak{f}_2(2\tau)}{\eta(2\tau)}g_{4j-4r-1,4r-1}\big(\frac{\tau}{2}\big), & \text{if } r+1\le j\le 2r-1.\end{cases}$$

So

$$V_j^*(\tau) = e^{\pi i\frac{(4r-4j+1)^2}{32r-8}}\frac{\mathfrak{f}_2(2\tau-1)}{\eta(2\tau-1)}g_{|4r-4j+1|,4r-1}\Big(\frac{2\tau-1}{4}\Big).$$

By means of (5.1) and (5.3), we have

$$V_j^*(\tau)=e^{\pi i\frac{(4r-4j+1)^2}{32r-8}}\frac{\mathfrak{f}_2(2\tau)}{\eta(2\tau)}g_{|4r-4j+1|,4r-1}\Big(\frac{2\tau-1}{4}\Big).$$

For $1\le j\le 2r-1$, let

$$W_j(\tau)=\frac{\mathfrak{f}_2(2\tau)}{\eta(2\tau)}g_{2j-1,4r-1}\Big(\frac{2\tau-1}{4}\Big), \tag{5.17}$$

and let $\boldsymbol{W}_{2r-1}(\tau)=(W_1(\tau),\dots,W_{2r-1}(\tau))^T$. Then

$$\boldsymbol{V}^*_{2r-1}(\tau)=P\,\boldsymbol{W}_{2r-1}(\tau), \tag{5.18}$$

where $P=(p_{j,k})_{(2r-1)\times(2r-1)}$, and

$$p_{j,k}=\begin{cases} e^{\pi i\frac{(4r-4j+1)^2}{32r-8}}, & \text{if } 1\le j\le 2r-1, k=2r-2j+1 \text{ or } 2j-2r,\\ 0, & \text{otherwise.}\end{cases}$$

It follows from (5.17) that

$$W_j\left(-\frac{1}{2\tau}\right)=\frac{\mathfrak{f}_2(-1/\tau)}{\eta(-1/\tau)}g_{2j-1,4r-1}\Big(-\frac{\tau+1}{4\tau}\Big).$$

Applying Lemma 5.2 and using (5.1) and (5.2), we have

$$\begin{aligned} &W_j\left(-\frac{1}{2\tau}\right)\\ &=\sqrt{\frac{i}{2(4r-1)}}\frac{\mathfrak{f}_1(\tau)}{\eta(\tau)}\sum_{\substack{1\le\ell\le 4r-2\\ \ell\text{ odd}}}\left(e^{\pi i\frac{2-2j-\ell-(2j+\ell-3)^2r}{8r-2}}+e^{\pi i\frac{2-2j+\ell-(2j-\ell-3)^2r}{8r-2}}\right)g_{\ell,4r-1}\Big(\frac{\tau+1}{4}\Big)\\ &=\sqrt{\frac{i}{2(4r-1)}}\sum_{\ell=1}^{2r-1}\frac{\mathfrak{f}_1(\tau)}{\eta(\tau)}\left(e^{-\pi i\frac{2j+2\ell-3+(2j+2\ell-4)^2r}{8r-2}}+e^{-\pi i\frac{2j-2\ell-1+(2j-2\ell-2)^2r}{8r-2}}\right)g_{2\ell-1,4r-1}\Big(\frac{\tau+1}{4}\Big). \end{aligned} \tag{5.19}$$

By (5.16), we obtain

$$\begin{aligned} U_j(\tau+1)&=\frac{\mathfrak{f}(\tau+1)}{\eta(\tau+1)}g_{2j-1,4r-1}\Big(\frac{\tau+1}{4}\Big)\\ &=e^{-\frac{\pi i}{8}}\frac{\mathfrak{f}_1(\tau)}{\eta(\tau)}g_{2j-1,4r-1}\Big(\frac{\tau+1}{4}\Big)(\text{by (5.1) and (5.3)}). \end{aligned} \tag{5.20}$$

It follows from (5.19) and (5.20) that

$$W_j(-\frac{1}{2\tau})=\frac{1}{\sqrt{8r-2}}e^{\frac{3\pi i}{8}}\sum_{\ell=1}^{2r-1}\left(e^{-\pi i\frac{2j+2\ell-3+(2j+2\ell-4)^2r}{8r-2}}+e^{-\pi i\frac{2j-2\ell-1+(2j-2\ell-2)^2r}{8r-2}}\right)U_\ell(\tau+1),$$

which leads to

$$\boldsymbol{W}_{2r-1}\Big(-\frac{1}{2\tau}\Big)=X\,\boldsymbol{U}_{2r-1}(\tau+1), \tag{5.21}$$

where $X=(x_{j,\ell})_{(2r-1)\times(2r-1)}$, and

$$x_{j,\ell}=\frac{1}{\sqrt{8r-2}}e^{\frac{3\pi i}{8}}(e^{-\pi i\frac{2j+2\ell-3+(2j+2\ell-4)^2r}{8r-2}}+e^{-\pi i\frac{2j-2\ell-1+(2j-2\ell-2)^2r}{8r-2}}).$$

In view of (5.11) and (5.12), we have

$$U_j^*(2\tau)=(-1)^{\frac{j(j-1)}{2}}e^{-\frac{2j^2-2j-2r+1}{16r-4}\pi i}U_j(2\tau+1).$$

Then

$$\boldsymbol{U}_{2r-1}^*(\tau)=\Lambda\,\boldsymbol{U}_{2r-1}(\tau+1), \tag{5.22}$$

where $\Lambda=(\lambda_{j,k})_{(2r-1)\times(2r-1)}$ and

$$\lambda_{j,k}=\begin{cases}0, & \text{if } j\neq k,\\ (-1)^{\frac{j(j-1)}{2}}e^{-\frac{2j^2-2j-2r+1}{16r-4}\pi i}, & \text{if } j=k.\end{cases}$$

Combining (5.18), (5.21) and (5.22) yields

$$\boldsymbol{V}_{2r-1}^*\Big(-\frac{1}{2\tau}\Big)=PX\Lambda^{-1}\,\boldsymbol{U}_{2r-1}^*(\tau). \tag{5.23}$$

Next we show that $PX\Lambda^{-1}=S$. By the definitions of P,X and Λ, for $1\le j\le r$, the (j,k)-entry of $PX\Lambda^{-1}$ is equal to

$$\begin{aligned}&\frac{1}{\sqrt{8r-2}}(-1)^{\frac{k(k-1)}{2}}e^{\frac{\pi i(4r-4j+1)^2}{32r-8}+\frac{3\pi i}{8}+\frac{\pi i(2k^2-2k-2r+1)}{16r-4}}\\ &\quad\cdot\Big(e^{-\pi i\frac{4r-4j+2k-1+4r(2r-2j+k-1)^2}{8r-2}}+e^{-\pi i\frac{4r-4j-2k+1+4r(2r-2j-k)^2}{8r-2}}\Big)\\ &=\sqrt{\frac{2}{4r-1}}\cos\frac{(2j-1)(2k-1)\pi}{2(4r-1)},\end{aligned}$$

and for $r+1\le j\le 2r-1$, the (j,k)-entry of $PX\Lambda^{-1}$ is equal to

$$\begin{aligned}&\frac{1}{\sqrt{8r-2}}(-1)^{\frac{k(k-1)}{2}}e^{\frac{\pi i(4r-4j+1)^2}{32r-8}+\frac{3\pi i}{8}+\frac{\pi i(2k^2-2k-2r+1)}{16r-4}}\\ &\quad\cdot\Big(e^{-\pi i\frac{4j-4r+2k-3+4r(2r-2j-k+2)^2}{8r-2}}+e^{-\pi i\frac{4j-4r-2k-1+4r(2r-2j+k+1)^2}{8r-2}}\Big)\\ &=\sqrt{\frac{2}{4r-1}}\cos\frac{(2j-1)(2k-1)\pi}{2(4r-1)}.\end{aligned}$$

Therefore, $PX\Lambda^{-1}=S$, so (5.15) holds. This completes the proof of Theorem 5.5. ∎

We also derived the following transformation formulas for $\boldsymbol{g}_{2r-1}(\tau)$ and $\boldsymbol{g}_{2r-1}^\vee(\tau)$.

Theorem 5.6. *For any integer r,*

$$\boldsymbol{g}_{2r-1}(\tau+2)=T\,\boldsymbol{g}_{2r-1}(\tau)\ \textit{and}\ \boldsymbol{g}_{2r-1}^\vee(\tau+1)=T^\vee\,\boldsymbol{g}_{2r-1}^\vee(\tau) \tag{5.24}$$

where $T=(t_{j,k})_{(2r-1)\times(2r-1)}, T^{\vee}=(t^{\vee}_{j,k})_{(2r-1)\times(2r-1)}$*, and*

$$t_{j,k}=\begin{cases}0, & \text{if } j\neq k,\\ e^{\frac{(4r-4j+1)^2}{8r-2}\pi i}, & \text{if } j=k,\end{cases}\qquad t^{\vee}_{j,k}=\begin{cases}0, & \text{if } j\neq k,\\ (-1)^{\frac{j(j-1)}{2}}e^{\frac{2j^2-2j-2r+1}{16r-4}\pi i}, & \text{if } j=k.\end{cases}$$

Moreover, if r *is odd, then*

$$\boldsymbol{g}_{2r-1}(\tau+1)=\widehat{T}\,\boldsymbol{g}_{2r-1}(\tau), \tag{5.25}$$

where $\widehat{T}=(\hat{t}_{j,k})_{(2r-1)\times(2r-1)}$*, and*

$$\hat{t}_{j,k}=\begin{cases}0, & \text{if } j\neq k,\\ e^{\frac{(4r-4j+1)^2}{16r-4}\pi i}, & \text{if } j=k.\end{cases}$$

Proof. By the definitions of $\boldsymbol{g}_{2r-1}(\tau)$ and $\boldsymbol{g}^{\vee}_{2r-1}(\tau)$, it is easy to see that for integer r and $0\leq j\leq 2r-1$,

$$g_j(\tau+2)=e^{\frac{(4r-4j+1)^2}{8r-2}\pi i}g_j(\tau),$$

$$g^{\vee}_j(2\tau+1)=(-1)^{\frac{j(j-1)}{2}}e^{\frac{2j^2-2j-2r+1}{16r-4}\pi i}g^{\vee}_j(2\tau),$$

which implies (5.24).

If r is odd, then

$$g_j(\tau+1)=e^{\frac{(4r-4j+1)^2}{16r-4}\pi i}g_j(\tau),$$

which implies (5.25). ∎

Now we can give a proof of Theorem 1.6.

Proof of Theorem 1.6. Let

$$M=\begin{pmatrix}\boldsymbol{O} & 2S\\ S & \boldsymbol{O}\end{pmatrix},\quad Q=\begin{pmatrix}T & \boldsymbol{O}\\ \boldsymbol{O} & (T^{\vee})^2\end{pmatrix},\quad \widehat{Q}=\begin{pmatrix}\widehat{T} & \boldsymbol{O}\\ \boldsymbol{O} & T^{\vee}\end{pmatrix},$$

where S is defined in Theorem 5.5, $T, T^{\vee}$ and $\widehat{T}$ are given in Theorem 5.6. From Theorem 5.5 and Theorem 5.6, we see that for $r\geq 2$,

$$\boldsymbol{G}_{4r-2}(\tau+2)=Q\,\boldsymbol{G}_{4r-2}(\tau),$$

$$\boldsymbol{G}_{4r-2}\left(\frac{\tau}{4\tau+1}\right)=\boldsymbol{G}_{4r-2}\left(-\frac{1}{-2(2+\frac{1}{2\tau})}\right)=M\,\boldsymbol{G}_{4r-2}\left(-2-\frac{1}{2\tau}\right)$$

$$=MQ^{-1}\,\boldsymbol{G}_{4r-2}\left(-\frac{1}{2\tau}\right)=MQ^{-1}M\,\boldsymbol{G}_{4r-2}(\tau).$$

Let Γ' be the group generated by $\gamma_1=\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$ and $\gamma_2=\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)$, and let the multiplicative function $\rho\colon\Gamma'\to GL_{4r-2}(\mathbb{C})$ satisfy that $\rho(\gamma_1)=Q$ and $\rho(\gamma_2)=MQ^{-1}M$. Hence $\boldsymbol{G}_{4r-2}(\tau)$ is a vector-valued automorphic form of multiplier ρ for Γ'.

If r is odd, then

$$\boldsymbol{G}_{4r-2}(\tau+1) = \widehat{Q}\ \boldsymbol{G}_{4r-2}(\tau), \tag{5.26}$$

$$\begin{aligned}\boldsymbol{G}_{4r-2}\left(\frac{\tau}{2\tau+1}\right) &= \boldsymbol{G}_{4r-2}\left(-\frac{1}{-2(1+\frac{1}{2\tau})}\right) = M\ \boldsymbol{G}_{4r-2}\left(-\frac{1}{2\tau}-1\right) \\ &= M\widehat{Q}^{-1}\ \boldsymbol{G}_{4r-2}\left(-\frac{1}{2\tau}\right) = M\widehat{Q}^{-1}M\ \boldsymbol{G}_{4r-2}(\tau). \end{aligned} \tag{5.27}$$

Notice that

$$\widehat{\gamma}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \widehat{\gamma}_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

are all generators of $\Gamma_0(2)$. Let the multiplicative function $\widehat{\rho}\colon \Gamma_0(2) \to GL_{4r-2}(\mathbb{C})$ satisfy that $\widehat{\rho}(\widehat{\gamma}_1) = \widehat{Q}$ and $\widehat{\rho}(\widehat{\gamma}_2) = M\widehat{Q}^{-1}M$. It follows from (5.26) and (5.27) that for any $\gamma \in \Gamma_0(2)$,

$$\boldsymbol{G}_{4r-2}(\gamma\tau) = \widehat{\rho}(\gamma)\boldsymbol{G}_{4r-2}(\tau).$$

Since for any $1 \le j \le 2r-1$, $g_j(\tau)$ and $g_j^\vee(\tau)$ can be expressed as sums of generalized eta-quotients, by [5, Lemma 2.6], for any $\gamma \in \Gamma$, there exists positive integers $N_{\gamma,j}$ and $N_{\gamma,j}^\vee$ such that

$$g_j(\gamma\tau) = \sum_n a_{n,j} q^{n/N_{\gamma,j}}$$

and

$$g_j^\vee(\gamma\tau) = \sum_n a_{n,j}^\vee q^{n/N_{\gamma,j}^\vee},$$

where $a_{n,j} = 0$ and $a_{n,j}^\vee = 0$ for $n \ll 0$. Therefore, $\boldsymbol{G}_{4r-2}(\tau)$ is a vector-valued modular function associated with $\widehat{\rho}$ for $\Gamma_0(2)$. ∎

5.3. **Proof of Theorem 1.7.** For $r \ge 2$ and $1 \le j \le 2r-2$, recall that

$$\begin{aligned} h_j(\tau) &= q^{\frac{(4j-4r+1)^2}{32r-24}} \left(q^{\frac{4j-2r-1}{4}} \frac{(q^{4(2r-1-j)}, q^{8r+4j-8}, q^{16r-12}; q^{16r-12})_\infty}{(q, q^3, q^4; q^4)_\infty} \right. \\ &\quad \left. + \frac{(q^{2(2r-j-1)}, -q^{2j-1}, -q^{4r-3}; -q^{4r-3})_\infty}{(q^2, q^2, q^4; q^4)_\infty} \right), \\ h_j^\vee(2\tau) &= \frac{1}{2} q^{\frac{j^2-j-r+1}{8r-6}} \left(\frac{(q^{2r-j-1}, q^{2r+j-2}, q^{4r-3}; q^{4r-3})_\infty}{(q, q^3, q^4; q^4)_\infty} \right. \\ &\quad \left. + \frac{((-q)^{2r-j-1}, (-q)^{2r+j-2}, -q^{4r-3}; -q^{4r-3})_\infty}{(-q, -q^3, q^4; q^4)_\infty} \right). \end{aligned}$$

Let

$$\boldsymbol{h}_{2r-2}(\tau) = (h_1(\tau), \dots, h_{2r-2}(\tau))^T, \quad \boldsymbol{h}_{2r-2}^\vee(\tau) = (h_1^\vee(\tau), \dots, h_{2r-2}^\vee(\tau))^T.$$

We derive the following transformation formula on the "Langlands dual" pair $\boldsymbol{h}_{2r-2}(\tau)$ and $\boldsymbol{h}^{\vee}_{2r-2}(\tau)$.

Theorem 5.7. *For $r \geq 2$,*

$$\boldsymbol{h}_{2r-2}\left(-\frac{1}{2\tau}\right) = 2\widetilde{S}\,\boldsymbol{h}^{\vee}_{2r-2}(\tau), \quad \boldsymbol{h}^{\vee}_{2r-2}\left(-\frac{1}{2\tau}\right) = \widetilde{S}\,\boldsymbol{h}_{2r-2}(\tau), \tag{5.28}$$

where $\widetilde{S} = (\widetilde{s}_{j,k})_{(2r-2)\times(2r-2)}$, and $\widetilde{s}_{j,k} = \sqrt{\frac{2}{4r-3}}\cos\frac{(2j-1)(2k-1)}{8r-6}\pi$.

Proof. Taking $N = 4r-3$ in Lemma 5.4, in order to prove (5.28), we only need to prove

$$\boldsymbol{h}_{2r-2}\left(-\frac{1}{2\tau}\right) = 2\widetilde{S}\,\boldsymbol{h}^{\vee}_{2r-2}(\tau), \tag{5.29}$$

For $1 \leq j \leq 2r-2$, let

$$\widetilde{V}_j(\tau) = q^{\frac{16j^2-16j-4r+7}{32r-24}}\frac{(q^{4(2r-1-j)}, q^{8r+4j-8}, q^{16r-12}; q^{16r-12})_\infty}{(q, q^3, q^4; q^4)_\infty}, \tag{5.30}$$

$$\widetilde{V}^*_j(\tau) = q^{\frac{(4j-4r+1)^2}{32r-24}}\frac{(q^{2(2r-j-1)}, -q^{2j-1}, -q^{4r-3}; -q^{4r-3})_\infty}{(q^2, q^2, q^4; q^4)_\infty}, \tag{5.31}$$

$$\widetilde{U}_j(2\tau) = q^{\frac{j^2-j-r+1}{8r-6}}\frac{(q^{2r-j-1}, q^{2r+j-2}, q^{4r-3}; q^{4r-3})_\infty}{(q, q^3, q^4; q^4)_\infty}, \tag{5.32}$$

$$\widetilde{U}^*_j(2\tau) = q^{\frac{j^2-j-r+1}{8r-6}}\frac{((-q)^{2r-j-1}, (-q)^{2r+j-2}, -q^{4r-3}; -q^{4r-3})_\infty}{(-q, -q^3, q^4; q^4)_\infty}. \tag{5.33}$$

Then

$$h_j(\tau) = \widetilde{V}_j(\tau) + \widetilde{V}^*_j(\tau), \quad h^{\vee}_j(\tau) = \frac{1}{2}(\widetilde{U}_j(\tau) + \widetilde{U}^*_j(\tau)). \tag{5.34}$$

Let

$$\widetilde{\boldsymbol{V}}_{2r-2}(\tau) = (\widetilde{V}_1(\tau), \ldots, \widetilde{V}_{2r-2}(\tau))^T,$$

$$\widetilde{\boldsymbol{V}}^*_{2r-2}(\tau) = (\widetilde{V}^*_1(\tau), \ldots, \widetilde{V}^*_{2r-2}(\tau))^T,$$

$$\widetilde{\boldsymbol{U}}_{2r-2}(\tau) = (\widetilde{U}_1(\tau), \ldots, \widetilde{U}_{2r-2}(\tau))^T,$$

and

$$\widetilde{\boldsymbol{U}}^*_{2r-2}(\tau) = (\widetilde{U}^*_1(\tau), \ldots, \widetilde{U}^*_{2r-2}(\tau))^T.$$

Then

$$\boldsymbol{h}_{2r-2}(\tau) = \widetilde{\boldsymbol{V}}_{2r-2}(\tau) + \widetilde{\boldsymbol{V}}^*_{2r-2}(\tau), \quad \boldsymbol{h}^{\vee}_{2r-2}(\tau) = \frac{1}{2}(\widetilde{\boldsymbol{U}}_{2r-2}(\tau) + \widetilde{\boldsymbol{U}}^*_{2r-2}(\tau)).$$

Hence, in order to prove (5.29), it suffices to prove

$$\widetilde{\boldsymbol{V}}_{2r-2}\left(-\frac{1}{2\tau}\right) = \widetilde{S}\,\widetilde{\boldsymbol{U}}_{2r-2}(\tau), \tag{5.35}$$

and

$$\widetilde{\boldsymbol{V}}^*_{2r-2}\left(-\frac{1}{2\tau}\right) = \widetilde{S}\,\widetilde{\boldsymbol{U}}^*_{2r-2}(\tau). \tag{5.36}$$

Proof of (5.35)**.** It follows from (5.30) and (5.31) that

$$\begin{aligned} \widetilde{V}_j(\tau) &= \frac{\mathfrak{f}(2\tau)}{\eta(2\tau)} g_{2j-1,4r-3}(2\tau), \\ \widetilde{U}_j(2\tau) &= \frac{\mathfrak{f}(2\tau)}{\eta(2\tau)} g_{2j-1,4r-3}\left(\frac{\tau}{2}\right). \end{aligned} \tag{5.37}$$

So,

$$\widetilde{V}_j\left(-\frac{1}{4\tau}\right) = \frac{\mathfrak{f}\left(-\frac{1}{2\tau}\right)}{\eta\left(-\frac{1}{2\tau}\right)} g_{2j-1,4r-3}\left(-\frac{1}{2\tau}\right).$$

By means of (5.1), (5.2) and Lemma 5.1, we have

$$\begin{aligned} &\widetilde{V}_j\left(-\frac{1}{4\tau}\right) \\ &= \frac{1}{\sqrt{8r-6}} \frac{\mathfrak{f}(2\tau)}{\eta(2\tau)} \sum_{k=0}^{8r-7} e^{\frac{(2j-1)(2k+1)}{8r-6}\pi i} h_{k+\frac{1}{2},4r-3}(2\tau) \\ &= \frac{1}{\sqrt{8r-6}} \frac{\mathfrak{f}(2\tau)}{\eta(2\tau)} \sum_{k=0}^{4r-4} \left(e^{\frac{(2j-1)(2k+1)}{8r-6}\pi i} h_{k+\frac{1}{2},4r-3}(2\tau) + e^{-\frac{(2j-1)(2k+1)}{8r-6}\pi i} h_{k+\frac{1}{2},4r-3}(2\tau) \right) \\ &= \sqrt{\frac{2}{4r-3}} \sum_{k=0}^{4r-4} \cos\frac{(2j-1)(2k+1)\pi}{8r-6} \frac{\mathfrak{f}(2\tau)}{\eta(2\tau)} h_{k+\frac{1}{2},4r-3}(2\tau) \\ &= \sqrt{\frac{2}{4r-3}} \sum_{k=0}^{2r-3} \cos\frac{(2j-1)(2k+1)\pi}{8r-6} \frac{\mathfrak{f}(2\tau)}{\eta(2\tau)} (h_{k+\frac{1}{2},4r-3}(2\tau) - h_{4r-k-\frac{7}{2},4r-3}(2\tau)). \end{aligned}$$

In view of (5.9), the above identity can be rewritten as

$$\begin{aligned} &\widetilde{V}_j\left(-\frac{1}{4\tau}\right) \\ &= \sqrt{\frac{2}{4r-3}} \sum_{k=0}^{2r-3} \cos\frac{(2j-1)(2k+1)\pi}{8r-6} \frac{\mathfrak{f}(2\tau)}{\eta(2\tau)} \left(h_{4k+2,16r-12}\left(\frac{\tau}{2}\right) - h_{16r-4k-14,16r-12}\left(\frac{\tau}{2}\right) \right) \\ &= \sqrt{\frac{2}{4r-3}} \sum_{k=0}^{2r-3} \cos\frac{(2j-1)(2k+1)\pi}{8r-6} \frac{\mathfrak{f}(2\tau)}{\eta(2\tau)} g_{2k+1,4r-3}\left(\frac{\tau}{2}\right) \\ &= \sqrt{\frac{2}{4r-3}} \sum_{k=1}^{2r-2} \cos\frac{(2j-1)(2k-1)\pi}{8r-6} \frac{\mathfrak{f}(2\tau)}{\eta(2\tau)} g_{2k-1,4r-3}\left(\frac{\tau}{2}\right) \\ &= \sqrt{\frac{2}{4r-3}} \sum_{k=1}^{2r-2} \cos\frac{(2j-1)(2k-1)\pi}{8r-6} \widetilde{U}_k(2\tau) \text{ (by (5.37))}. \end{aligned}$$

Thus,

$$\widetilde{V}_j\left(-\frac{1}{2\tau}\right)=\sqrt{\frac{2}{4r-3}}\sum_{k=1}^{2r-2}\cos\frac{(2j-1)(2k-1)\pi}{8r-6}\widetilde{U}_k(\tau), \tag{5.38}$$

which implies (5.35).

Proof of (5.36)**.** By (5.31), it is easy to see that

$$\begin{aligned}\widetilde{V}^*_j\left(\tau+\frac{1}{2}\right)&=e^{\frac{(4r-4j-1)^2}{32r-24}\pi i}q^{\frac{(4r-4j-1)^2}{32r-24}}\frac{(q^{2(2r-j-1)},q^{2j-1},q^{4r-3};q^{4r-3})_\infty}{(q^2,q^2,q^4;q^4)_\infty}\\&=\begin{cases}e^{\frac{(4r-4j-1)^2}{32r-24}\pi i}\frac{\mathfrak{f}_2(2\tau)}{\eta(2\tau)}g_{4r-4j-1,4r-3}\left(\frac{\tau}{2}\right), & \text{if } 1\le j\le r-1,\\ e^{\frac{(4r-4j-1)^2}{32r-24}\pi i}\frac{\mathfrak{f}_2(2\tau)}{\eta(2\tau)}g_{4j-4r+1,4r-3}\left(\frac{\tau}{2}\right), & \text{if } r\le j\le 2r-2.\end{cases}\end{aligned}$$

So,

$$\begin{aligned}\widetilde{V}^*_j(\tau)&=e^{\frac{(4r-4j-1)^2}{32r-24}\pi i}\frac{\mathfrak{f}_2(2\tau-1)}{\eta(2\tau-1)}g_{|4r-4j-1|,4r-3}\left(\frac{2\tau-1}{4}\right)\\&=e^{\frac{(4r-4j-1)^2}{32r-24}\pi i}\frac{\mathfrak{f}_2(2\tau)}{\eta(2\tau)}g_{|4r-4j-1|,4r-3}\left(\frac{2\tau-1}{4}\right)\ \text{(by (5.1) and (5.3))}.\end{aligned}$$

Let

$$\widetilde{W}_j(\tau)=\frac{\mathfrak{f}_2(2\tau)}{\eta(2\tau)}g_{2j-1,4r-3}\left(\frac{2\tau-1}{4}\right), \tag{5.39}$$

and $\widetilde{\boldsymbol{W}}_{2r-2}(\tau)=(\widetilde{W}_1(\tau),\ldots,\widetilde{W}_{2r-2}(\tau))^T$. Then

$$\widetilde{\boldsymbol{V}}^*_{2r-2}(\tau)=\widetilde{P}\,\widetilde{\boldsymbol{W}}_{2r-2}(\tau), \tag{5.40}$$

where $\widetilde{P}=(\widetilde{p}_{j,k})_{(2r-2)\times(2r-2)}$,

$$\widetilde{p}_{j,k}=\begin{cases}e^{\frac{(4r-4j-1)^2}{32r-24}\pi i}, & \text{if } 1\le j\le 2r-2, k=2r-2j, \text{or } k=2j-2r+1,\\ 0, & \text{otherwise.}\end{cases}$$

By means of (5.39), we have

$$\begin{aligned}&\widetilde{W}_j\left(-\frac{1}{2\tau}\right)\\&=\frac{\mathfrak{f}_2\left(-\frac{1}{\tau}\right)}{\eta\left(-\frac{1}{\tau}\right)}g_{2j-1,4r-3}\left(-\frac{\tau+1}{4\tau}\right)\\&=\frac{1}{\sqrt{2i(4r-3)}}\frac{\mathfrak{f}_1(\tau)}{\eta(\tau)}\\&\quad\sum_{\substack{1\le\ell\le 4r-4\\ \ell\text{ odd}}}\left(e^{\frac{2-2j-\ell-(2j+\ell-3)^2(3r-2)}{8r-6}\pi i}+e^{\frac{2-2j+\ell-(2j-\ell-3)^2(3r-2)}{8r-6}\pi i}\right)g_{\ell,4r-3}\left(\frac{\tau+1}{4}\right)\end{aligned}$$

$$= \frac{1}{\sqrt{2i(4r-3)}} \frac{\mathfrak{f}_1(\tau)}{\eta(\tau)}$$

$$\sum_{\ell=1}^{2r-2} \left(e^{-\frac{2j+2\ell-3+(2j+2\ell-4)^2(3r-2)}{8r-6}\pi i} + e^{-\frac{2j-2\ell-1+(2j-2\ell-2)^2(3r-2)}{8r-6}\pi i} \right) g_{2\ell-1,4r-3}\left(\frac{\tau+1}{4}\right)$$

$$= \frac{1}{\sqrt{2(4r-3)}} e^{-\frac{\pi i}{8}} \sum_{\ell=1}^{2r-2} \left(e^{-\frac{2j+2\ell-3+(2j+2\ell-4)^2(3r-2)}{8r-6}\pi i} + e^{-\frac{2j-2\ell-1+(2j-2\ell-2)^2(3r-2)}{8r-6}\pi i} \right) \widetilde{U}_j(\tau+1),$$

where the last identity follows from

$$\widetilde{U}_j(\tau+1) = \frac{\mathfrak{f}(\tau+1)}{\eta(\tau+1)} g_{2j-1,4r-3}\left(\frac{\tau+1}{4}\right)$$
$$= e^{-\frac{\pi i}{8}} \frac{\mathfrak{f}_1(\tau)}{\eta(\tau)} g_{2j-1,4r-3}\left(\frac{\tau+1}{4}\right) \text{ (by (5.1) and (5.3))}.$$

So

$$\widetilde{\boldsymbol{W}}_{2r-2}\left(-\frac{1}{2\tau}\right) = \widetilde{X}\, \widetilde{\boldsymbol{U}}_{2r-2}(\tau+1), \tag{5.41}$$

where $\widetilde{X} = (\widetilde{x}_{j,\ell})_{(2r-2)\times(2r-2)}$ and

$$\widetilde{x}_{j,\ell} = \frac{1}{\sqrt{2(4r-3)}} e^{-\frac{\pi i}{8}} \left(e^{-\frac{2j+2\ell-3+(2j+2\ell-4)^2(3r-2)}{8r-6}\pi i} + e^{-\frac{2j-2\ell-1+(2j-2\ell-2)^2(3r-2)}{8r-6}\pi i} \right).$$

By (5.32) and (5.33), we have

$$\widetilde{U}_j(2\tau+1) = e^{\frac{j^2-j-r+1}{8r-6}\pi i} \widetilde{U}_j^*(2\tau),$$

and so

$$\widetilde{U}_j(\tau+1) = e^{\frac{j^2-j-r+1}{8r-6}\pi i} \widetilde{U}_j^*(\tau).$$

Thus,

$$\widetilde{\boldsymbol{U}}_{2r-2}(\tau+1) = \widetilde{\Lambda}\, \widetilde{\boldsymbol{U}}^*_{2r-2}(\tau), \tag{5.42}$$

where $\widetilde{\Lambda} = (\widetilde{\lambda}_{j,k})_{(2r-2)\times(2r-2)}$, and

$$\widetilde{\lambda}_{j,k} = \begin{cases} e^{\frac{j^2-j-r+1}{8r-6}\pi i}, & \text{if } j=k, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, from (5.40), (5.41) and (5.42), we have

$$\widetilde{\boldsymbol{V}}^*_{2r-2}\left(-\frac{1}{2\tau}\right) = \widetilde{P}\widetilde{X}\widetilde{\Lambda}\, \widetilde{\boldsymbol{U}}^*_{2r-2}(\tau).$$

For $1 \le j \le r-1$, the (j,k)-entry of $\widetilde{P}\widetilde{X}\widetilde{\Lambda}$ is equal to

$$\frac{1}{\sqrt{2(4r-3)}} e^{-\frac{\pi i}{8}+\frac{(4r-4j-1)^2}{32r-24}\pi i+\frac{k^2-k-r+1}{8r-6}\pi i} \left(e^{-\frac{2(2r-2j)+2k-3+(2(2r-2j)+2k-4)^2(3r-2)}{8r-6}\pi i}\right.$$

$$+e^{-\frac{2(2r-2j)-2k-1+(2(2r-2j)-2k-2)^2(3r-2)}{8r-6}\pi i}\Big)$$

$$=\sqrt{\frac{2}{4r-3}}\cos\frac{(2j-1)(2k-1)}{2(4r-3)}\pi,$$

and for $r \le j \le 2r-2$, the (j,k)-entry of $\widetilde{P}\widetilde{X}\widetilde{\Lambda}$ is equal to

$$\frac{1}{\sqrt{2(4r-3)}}e^{-\frac{\pi i}{8}+\frac{(4r-4j-1)^2}{32r-24}\pi i+\frac{k^2-k-r+1}{8r-6}\pi i}\Big(e^{-\frac{2(2j-2r+1)+2k-3+(2(2j-2r+1)+2k-4)^2(3r-2)}{8r-6}\pi i}$$

$$+e^{-\frac{2(2j-2r+1)-2k-1+(2(2j-2r+1)-2k-2)^2(3r-2)}{8r-6}\pi i}\Big)$$

$$=\sqrt{\frac{2}{4r-3}}\cos\frac{(2j-1)(2k-1)}{2(4r-3)}\pi,$$

so $\widetilde{P}\widetilde{X}\widetilde{\Lambda}=\widetilde{S}$. Thus, (5.36) holds. ∎

Theorem 5.8. *For any $r \ge 2$,*

$$\boldsymbol{h}_{2r-2}(\tau+4)=\widetilde{T}\,\boldsymbol{h}_{2r-2}(\tau),\quad \boldsymbol{h}^{\vee}_{2r-2}(\tau+1)=\widetilde{T}^{\vee}\,\boldsymbol{h}^{\vee}_{2r-2}(\tau),\tag{5.43}$$

where $\widetilde{T}=(\widetilde{t}_{j,k})_{(2r-2)\times(2r-2)}$, $\widetilde{T}^{\vee}=(\widetilde{t}^{\vee}_{j,k})_{(2r-2)\times(2r-2)}$,

$$\widetilde{t}_{j,k}=\begin{cases} e^{\frac{(4j-4r+1)^2}{4r-3}\pi i}, & \text{if } j=k,\\ 0, & \text{otherwise,}\end{cases}\qquad \widetilde{t}^{\vee}_{j,k}=\begin{cases} e^{\frac{j^2-j-r+1}{8r-6}\pi i}, & \text{if } j=k,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. It is obvious that for $1 \le j \le 2r-2$,

$$h_j(\tau+4)=e^{\frac{(4j-4r+1)^2}{4r-3}\pi i}h_j(\tau),$$

$$h^{\vee}_j(2\tau+1)=e^{\frac{j^2-j-r+1}{8r-6}\pi i}h^{\vee}_j(2\tau).$$

The the proof is complete. ∎

Proof of Theorem 1.7. Let

$$\widetilde{M}=\begin{pmatrix}\boldsymbol{O} & 2\widetilde{S}\\ \widetilde{S} & \boldsymbol{O}\end{pmatrix},\quad \widetilde{Q}=\begin{pmatrix}\widetilde{T} & \boldsymbol{O}\\ \boldsymbol{O} & (\widetilde{T}^{\vee})^4\end{pmatrix},$$

where $\widetilde{S}$ is given in Theorem 5.7, $\widetilde{T}$ and $\widetilde{T}^{\vee}$ are defined in Theorem 5.8. By Theorem 5.7 and Theorem 5.8, we have

$$\boldsymbol{H}_{4r-4}(\tau+4)=\widetilde{Q}\;\boldsymbol{H}_{4r-4}(\tau),$$

$$\boldsymbol{H}_{4r-4}\left(\frac{\tau}{8\tau+1}\right)=\boldsymbol{H}_{4r-4}\left(-\frac{1}{-2(4+\frac{1}{2\tau})}\right)=\widetilde{M}\;\boldsymbol{H}_{4r-4}\left(-4-\frac{1}{2\tau}\right)$$

$$=\widetilde{M}\widetilde{Q}^{-1}\;\boldsymbol{H}_{4r-4}\left(-\frac{1}{2\tau}\right)=\widetilde{M}\widetilde{Q}^{-1}\widetilde{M}\;\boldsymbol{H}_{4r-4}(\tau).$$

Let Γ' be the group generated by $\gamma_1 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} 1 & 0 \\ 8 & 1 \end{pmatrix}$ and the multiplicative function $\widetilde{\rho}\colon \Gamma' \to GL_{4r-4}(\mathbb{C})$ satisfy that $\widetilde{\rho}(\gamma_1) = \widetilde{Q}$ and $\widetilde{\rho}(\gamma_2) = \widetilde{M}\widetilde{Q}^{-1}\widetilde{M}$. Then for any $\gamma \in \Gamma'$,

$$\boldsymbol{H}_{4r-4}(\gamma\tau) = \widetilde{\rho}(\gamma)\boldsymbol{H}_{4r-4}(\tau),$$

which implies that $\boldsymbol{H}_{4r-4}(\tau)$ is a vector-valued automorphic form of the multiplier $\widetilde{\rho}$ for Γ'. ∎

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